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ON PSEUDO BE-ALGEBRAS

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Abstract

In this paper, we introduce the notion of pseudo BE-algebra which is a generalization of BE-algebra. We define the concepts of pseudo subalgebras and pseudo filters and prove that, under some conditions, pseudo subalgebra can be a pseudo filter. We prove that every homomorphic image and pre-image of a pseudo filter is also a pseudo filter. Furthermore, the notion of pseudo upper sets in pseudo BE-algebras introduced and is proved that every pseudo filter is an union of pseudo upper sets.

Keywords: BE-algebra, Pseudo BE-algebra, pseudo filter, pseudo upper set.

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1. Introduction

In 1966, Y. Imai and K. Iseki [8] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Some recent researchers led to generalizations of the notion of pseudo structure on some types of algebras. G Georgescu and A. Iorgulescu [5], and independently J. Rachunek [13], introduced pseudo-MV algebras which are a non-commutative generalization of MV-algebras. After pseudo-MV algebras, the pseudo-BL algebras [6], and the pseudo-BCK algebras as an extended notion of BCK-algebras by G. Georgescu and A. Iorgulescu [7], were introduced and studied. A. Walendziak give a system of axioms defining pseudo-BCK algebras [17]. In [9], Y.B. Jun and *et al.*, introduced the concepts of pseudo-atoms, pseudo-BCI ideals and pseudo-BCI homomorphisms in pseudo -BCI algebras and characterizations of a pseudo-BCI ideal, and provide conditions for a subset to be a pseudo-BCI ideal. Y.H. Kim and K.S. So [11], discuss on minimal elements in pseudo-BCI algebras.

The notion of BE-algebras was introduced by H.S. Kim and Y.H. Kim [10]. Using the notion of upper sets they gave an equivalent condition of the filter in BE-algebras. BE-algebras was deeply studied by S.S. Ahn and *et al.*, [1, 2, 3], Walendziak [16], A. Rezaei and *et al.*, [4, 14, 15]. Also, B.L. Meng [12], give a procedure by which generate a filter by a subset in a transitive BE-algebra, and give some characterizations of Noetherian and Artinian BE-algebras.

In the present paper we continue study in BE-algebras and introduce the notion of pseudo BE-algebras and pseudo filter in pseudo BE-algebras, also we discuss on the relationship between the pseudo sub algebras with pseudo filters in pseudo BE-algebras. We show that every pseudo filter is a pseudo subalgebra and give an example such that the converse is not true in general. The notion of pseudo upper set in pseudo BE-algebras is introduced and we show that every pseudo filter F of X is a union of pseudo upper sets (i.e., $F = \bigcup_{x\in F} A(x, 1)$).

Now, we review the definitions and some elementary aspects that are necessary for this paper.

Recall that a BE-algebra is an algebra $(X; *, 1)$ of type $(2, 0)$ satisfying the following axioms: ([10])

$(BE1)$ $x * x = 1$,

$(BE2)$ $x * 1 = 1$,

$(BE3)$ $1 * x = x$,

$(BE4)$ $x * (y * z) = y * (x * z)$, for all $x, y, z \in X$.

A binary relation "$\leq$" on BE-algebra X is defined by $x \leq y$ if and only if $x*y = 1$.

A BE-algebra X has the following properties:

(i) $x * (y * x) = 1$,

(ii) $y * ((y * x) * x) = 1$, for all $x, y \in X$.

A non-empty subset S of a BE-algebra X is called a subalgebra of X if $x * y \in S$ whenever $x, y \in S$.

A mapping $f : X \to Y$ of BE-algebras is called a homomorphism if $f(x*y) = f(x) * f(y)$, for all $x, y \in X$.

A non-empty subset F of BE-algebra X is called a filter of X if (1) $1 \in F$, (2) $x \in F$ and $x * y \in F$ implies $y \in F$.

Let A be a non-empty subset of BE-algebra X, then the set

$$< A >= \bigcap \{G \in F(X) |\ A \subseteq G\}$$

is called the filter generated by A, written $< A >$. If $A = \{a\}$, we will denote $< \{a\} >$, briefly by $< a >$, and we call it a principal filter of BE-algebra X. For $F \in F(X)$ and $a \in X$, we denote by F_a the filter generated by $F \cup \{a\}$, where $F(X)$ is the set of all filters of X.

2. Pseudo BE-algebras

Definition. An algebra $(X; *, \diamond, 1)$ of type $(2, 2, 0)$ is called a pseudo BE-algebra if satisfies in the following axioms:

$(pBE1)$ $x * x = 1$ and $x \diamond x = 1$,

$(pBE2)$ $x * 1 = 1$ and $x \diamond 1 = 1$,

$(pBE3)$ $1 * x = x$ and $1 \diamond x = x$,

$(pBE4)$ $x * (y \diamond z) = y \diamond (x * z)$,

$(pBE5)$ $x * y = 1 \Leftrightarrow x \diamond y = 1$,

for all $x, y, z \in X$.

In a pseudo BE-algebra, we can introduce a binary relation " $\leq$ " by $x \leq y \Leftrightarrow x * y = 1 \Leftrightarrow x \diamond y = 1$, for all $x, y \in X$.

Remark. If X is a pseudo BE-algebra satisfying $x * y = x \diamond y$, for all $x, y \in X$, then X is a BE-algebra.

Example 1. (i) Let $X = \{1, a, b, c\}$. Define the operations " $*$ " and " $\diamond$ " as follow:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	a	1	c
c	1	b	1	1

$\diamond$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	c	1	c
c	1	c	1	1

Then $(X; *, \diamond, 1)$ is a pseudo BE-algebra. We can see that $(X; *, 1)$ and $(X; \diamond, 1)$ are BE-algebras.

(ii) Let $X = \{1, a, b, c\}$. Define the operations " $*$" and " $\diamond$" on X as follow:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	1	b
b	1	a	1	c
c	1	1	1	1

$\diamond$	1	a	b	c
1	1	a	b	c
a	1	1	1	a
b	1	a	1	a
c	1	1	1	1

Then $(X; *, \diamond, 1)$ is a pseudo BE-algebra. Also, we can see that

$$a * (b * c) = a * c = b \neq b * (a * c) = b * b = 1.$$

Thus $(X; *, 1)$ is not a BE-algebra.

(iii) Let $X = (-\infty, -3] \cup \{-2, -1, 0, 1\}$. Define the operations " $*$" and " $\diamond$" on X as follows:

$$x * y = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{if } (x = 0 \text{ and } y = -2) \text{ or } (x = -1 \text{ and } y = -2) \\ y & \text{otherwise} \end{cases}$$

and

$$x \diamond y = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{if } x = -1 \text{ and } y = -2 \\ y & \text{otherwise.} \end{cases}$$

Then $(X; *, \diamond, 1)$ is a pseudo BE-algebra.

Proposition 2. *In a pseudo BE-algebra X, the following holds:*

(1) $x * (y \diamond x) = 1$, $x \diamond (y * x) = 1$,
(2) $x \diamond (y \diamond x) = 1$, $x * (y * x) = 1$,
(3) $x \diamond ((x \diamond y) * y) = 1$, $x * ((x * y) \diamond y) = 1$,
(4) $x * ((x \diamond y) * y) = 1$, $x \diamond ((x * y) \diamond y) = 1$,
(5) *if* $x \leq y * z$, *then* $y \leq x \diamond z$,
(6) *if* $x \leq y \diamond z$, *then* $y \leq x * z$,
(7) $1 \leq x$, *implies* $x = 1$,
(8) *if* $x \leq y$, *then* $x \leq z * y$ *and* $x \leq z \diamond y$,
(9) *if* $x * y = z$, *then* $y * z = y \diamond z = 1$ (*If* $x \diamond y = z$, *then* $y * z = y \diamond z = 1$),
(10) *if* $x * y = x$ *and* $x \neq 1$, *then* $x \diamond y \neq y$,
(11) *if* $x * y = y$ *and* $x \neq 1$, *then* $x \diamond y \neq x$,
(12) *if* $x * y = x$ *and* $x \diamond y = z$, *then* $x * z = x \diamond z = 1$, *and*

$$x * (y * z) = (x * y) * (x * z) = x \diamond (y * z) = (x \diamond y) * (x \diamond z) = 1,$$

(13) *if* $x * y = y$ *and* $x \diamond y = z$, *then* $x * z = z$ *and*

$$x * (y * z) = (x * y) * (x * z) = x * (y \diamond z) = (x * y) \diamond (x * z) = 1,$$

(14) *if* $x * y = z$ *and* $x \diamond y = t$, *then* $x \diamond z = x * t$,

for all $x, y, z \in X$.

Proof. (1) Let $x, y \in X$. By $(pBE4)$, $(pBE1)$ and $(pBE2)$ we have

$$x * (y \diamond x) = y \diamond (x * x) = y \diamond 1 = 1$$

and

$$x \diamond (y * x) = y * (x \diamond x) = y * 1 = 1.$$

(2) By $(pBE5)$ and (1) the proof is clear.

(3) Let $x, y \in X$. By $(pBE4)$ and $(pBE1)$ we have

$$x \diamond ((x \diamond y) * y) = (x \diamond y) * (x \diamond y) = 1$$

and

$$x * ((x * y) \diamond y) = (x * y) \diamond (x * y) = 1.$$

(4) By $(pBE5)$ and (3) the proof is clear.

(5) Let $x \leq y * z$ and $x, y, z \in X$. Then by $(pBE5)$ and $(pBE4)$ we have

$$x \diamond (y * z) = y * (x \diamond z) = 1.$$

Hence $y \leq x \diamond z$.

(6) The proof is similar to (5).

(7) Since $1 \leq x$, we have $1 * x = 1 \diamond x = 1$. Now, by $(pBE3)$, $x = 1 * x = 1 \diamond x = 1$.

(8) Let $z \in X$. Since $x * y = x \diamond y = 1$, by using $(pBE4)$, $(pBE2)$ and $(pBE5)$ we have

$$x \diamond (z * y) = z * (x \diamond y) = z * 1 = 1 \textit{ and so } x * (z * y) = 1.$$

Thus $x \leq z * y$. By a similar way, $x \leq z \diamond y$.

(9) Let $x * y = z$. By $(pBE4)$ and $(pBE1)$,

$$y \diamond z = y \diamond (x * y) = x * (y \diamond y) = x * 1 = 1.$$

Now, by $(pBE5)$, we get that $y * z = 1$.

(10) Let $x * y = x$ and $x \neq 1$. If $x \diamond y = y$, then

$$x * (x \diamond y) = x * y = x, \; x \diamond (x * y) = x \diamond x = 1,$$

and so by $(pBE4)$, $x = 1$, which is a contradiction.

(11) Let $x * y = y$ and $x \neq 1$. If $x \diamond y = x$, then

$$x * (x \diamond y) = x * x = 1, \; x \diamond (x * y) = x \diamond y = x,$$

and so by $(pBE4)$, $x = 1$, which is a contradiction.

(12) Let $x * y = x$ and $x \diamond y = z$. Then

$$x * (x \diamond y) = x * z, \; x \diamond (x * y) = x \diamond x = 1.$$

Hence by $(pBE4)$, we get that $x * z = 1$. Since $x \diamond y = z$, by (9), $y * z = y \diamond z = 1$. By $(pBE5)$, $x * z = x \diamond z = 1$. Then

$$x * (y * z) = x * 1 = 1, \; (x * y) * (x * z) = x * 1 = 1.$$

Hence by $(pBE5)$, $x \diamond (y * z) = 1$. Moreover, $(x \diamond y) * (x \diamond z) = z * 1 = 1$.

(13) Let $x * y = y$ and $x \diamond y = z$. By (9), $y * z = y \diamond z = 1$. Then

$$x * (x \diamond y) = x * z, \quad x \diamond (x * y) = x \diamond y = z.$$

Hence by $(pBE4)$, $x * z = z$. Then

$$x * (y \diamond z) = x * 1 = 1,\ x * (y * z) = x * 1 = 1,\ (x * y) * (x * z) = y * z = 1,$$

and so by $(pBE5)$, $(x * y) \diamond (x * z) = 1$.

(14) If $x * y = z$ and $x \diamond y = t$, then $x \diamond z = x \diamond (x * y) = x * (x \diamond y) = x * t$. ■

From now on X is a pseudo BE-algebra, unless otherwise is stated.

Definition. A subalgebra of X is a non-empty subset S of X which satisfies $x * y \in S$ and $x \diamond y \in S$, for all $x, y \in S$.

Example 3. Let $X = \{1, a, b, c, d\}$. Define operations " $*$ " and " $\diamond$ " on X as follows:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	a	1	a
b	1	1	1	1	a
c	1	a	a	1	a
d	1	1	1	1	1

$\diamond$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	c	1	c
b	1	1	1	1	c
c	1	a	b	1	d
d	1	1	1	1	1

Then $(X, *, \diamond)$ is a pseudo BE-algebra and $\{1, a, b, c\}$ is a sub-algebra of X.

Theorem 4. *If $\{S_i\}_{i\in I}$ is a family of subalgebras of X, then $\bigcap_{i\in I} S_i$ is a subalgebra of X, too.*

Definition. A non-empty subset F of X is called a pseudo filter of X if it satisfies in the following axioms:

$(pF1)$ $1 \in F$,

$(pF2)$ $x \in F$ and $x * y \in F$ imply $y \in F$.

A pseudo filter F is proper if and only if $F \neq X$.

A maximal pseudo filter is a proper filter such that it is not included in any other proper pseudo filter.

Proposition 5. *Let $F \subseteq X$ and $1 \in F$. F is a pseudo filter if and only if $x \in F$ and $x \diamond y \in F$ imply $y \in F$, for all $x, y \in X$ $(pF3)$.*

Proof. Let $x \in F$ and $x \diamond y \in F$. By Proposition 2 (4), we have $x * ((x \diamond y) * y) = 1$. By $(pF1)$, $x * ((x \diamond y) * y) \in F$. By assumption, $(x \diamond y) * y \in F$. Hence $y \in F$.

Conversely, let $x \in F$ and $x * y \in F$. By Proposition 2 (4), we have $x \diamond ((x * y) \diamond y) = 1$. By $(pF1)$, $x \diamond ((x * y) \diamond y) \in F$. From $(pF3)$ it follows that $(x * y) \diamond y \in F$. Hence $y \in F$. ■

Example 6. (i) $\{1\}$ and X are pseudo filters of X, in which we call trivial pseudo filters.

(ii) In Example 1(ii), $F = \{1, b\}$ is a pseudo filter, but $G = \{1, a\}$ is not a pseudo filter, because $a * c = 1 \in G$ and $a \in G$ but $c \notin G$.

Proposition 7. *Let F be a pseudo filter of X and $y \in X$. If $x \leq y$ and $x \in F$, then $y \in F$.*

Theorem 8. *Any pseudo filter of X is a pseudo subalgebra.*

Proof. Assume that F is a pseudo filter of X and $x, y \in F$. Since $x \leq y * x$, and by Proposition 7 we get that $y * x, y \diamond x \in F$. Hence F is a pseudo subalgebra of X. ■

Note. The converse of Theorem 8, is not true in general. Because in Example 1(ii), $S = \{1, a\}$ is a pseudo subalgebra, but it is not a pseudo filter.

Theorem 9. *If $\{F_i\}_{i \in I}$ is a family of pseudo filters of X, then $\bigcap_{i \in I} F_i$ is a pseudo filter of X, too.*

Proposition 10. *If F is a pseudo filter of X, then*

(i) $\forall x, y, z \in X, x, y \in F, x \leq y * z \Rightarrow z \in F$,

(ii) $\forall x, y, z \in X, x, y \in F, x \leq y \diamond z \Rightarrow z \in F$.

Proof. Assume that F is a pseudo filter of X and $x, y, z \in X$ such that $x, y \in F$ and $x \leq z * y$. Then $x \diamond (y * z) = 1 \in F$ and so $y * z \in F$. Since $y \in F$, $z \in F$.

Now, let $x, y, z \in X$ such that $x, y \in F$ and $x \leq y \diamond z$. Then $x * (y \diamond z) = 1 \in F$, and so $y \diamond z \in F$. Since $y \in F$, $z \in F$. ■

Theorem 11. *Let F be a subset of X containing 1. Then F is pseudo filter of X if and only if $x \leq y * z$ imply $z \in F$, for all $x, y \in F$ and $x \in X$.*

Proof. $(\Rightarrow)$. By Proposition 10, the proof is clear.
$(\Leftarrow)$. By assumption, $1 \in F$. Let $x, x \diamond y \in F$ and $y \in X$. Proposition 2(3) implies that $x \diamond ((x \diamond y) * y) = 1$ and hence $x \leq (x \diamond y) * y$. Since $x, x \diamond y \in F$, we conclude that $y \in F$. By Proposition 5, we get that F is a pseudo filter. ■

For element a of X, put

$$H(a) := \{x \in X : a \leq x\},$$

which is called the terminal section of an element a. Since $1, a \in H(a)$, we can see that $H(a)$ is not an empty set.

Theorem 12. *For any element a of X, the terminal section $H(a)$ is a pseudo filter if and only if the following implications hold:*

(i) $\forall x, y, z \in X, \quad z \leq x * y,$ *and* $z \leq x \Rightarrow z \leq y$;

(ii) $\forall x, y, z \in X, \quad z \leq x \diamond y,$ *and* $z \leq x \Rightarrow z \leq y$.

Proof. Assume that for each $a \in X$, $H(a)$ is a pseudo filter of X. Let $x, y, z \in X$ be such that $z \leq x * y$, $z \leq x \diamond y$, and $z \leq x$. Then $x * y \in H(z)$, $x \diamond y \in H(z)$ and $x \in H(z)$. Since $H(z)$ is pseudo filter, $y \in H(z)$. Therefore, $z \leq y$.

Conversely, consider $H(z)$, for $z \in X$. Obviously $1 \in H(z)$. Let $x * y \in H(z)$, for all $x \in H(z)$, i.e., $z \leq x * y$, and $z \leq x$. Then from hypothesis, $z \leq y$ i.e., $y \in H(z)$. Hence $H(z)$ is a pseudo filter of X, for all $z \in X$. ■

In the following, we provide some conditions for a subalgebra to be a pseudo filter.

Theorem 13. *Let F be a subalgebra of X. Then F is a pseudo filter of X if and only if for all $x, y \in X$,*

$$x \in F, \ y \in X \setminus F \Rightarrow x * y \in X \setminus F \text{ and } x \diamond y \in X \setminus F.$$

Proof. Assume that F is a pseudo filter of X and $x, y \in X$, such that $x \in F$ and $y \in X \setminus F$. If $x * y \notin X \setminus F$, then $x * y \in F$, i.e., $y \in F$, which is a contradiction. Hence $x * y \in X \setminus F$. Now, if $x \diamond y \notin X \setminus F$, then $x \diamond y \in F$, i.e., $y \in F$, which is a contradiction. Hence $x \diamond y \in X \setminus F$.

Conversely, assume that the hypothesis is valid. Since F is a subalgebra, then $1 \in F$. For every $x \in F$, let $x * y \in F$. If $y \notin F$, then $x * y \in X \setminus F$ by assumption, which is a contradiction. Hence $y \in F$. Therefore, F is a pseudo filter. ■

Definition. A mapping $f : X \to Y$ of pseudo BE-algebras is called a pseudo homomorphism if $f(x*y) = f(x)*f(y)$ and $f(x \diamond y) = f(x) \diamond f(y)$, for all $x, y \in X$.

Note that if $f : X \to Y$ is a pseudo homomorphism, then $f(1_X) = 1_Y$.

Theorem 14. *Let $f : X \to Y$ be a pseudo homomorphism. Then*

(i) *if F is a pseudo filter of Y, then $f^{-1}(F)$ is a pseudo filter of X,*

(ii) *if f is surjective and G is a pseudo filter of X containing* $\ker(f)$*, then* $f(G)$ *is a pseudo filter of* Y.

Proof. (i) Assume that F is a pseudo filter of Y. Obviously $1_X \in f^{-1}(F)$. Let $x, x*y \in f^{-1}(F)$. It follows that $f(x)*f(y) = f(x*y) \in F$. Then $f(y) \in F$, since $f(x) \in F$ and F is a pseudo filter of Y. Therefore, $y \in f^{-1}(F)$ and hence $f^{-1}(F)$ is a pseudo filter of X.

(ii) Obviously $1_Y \in f(G)$. Let $x, x*y \in f(G)$ and $y \in Y$. Since f is surjective, there exists $c \in X$ such that $f(c) = y$. Let $a, b \in G$ be such that $f(a) = x$ and $f(b) = x*y$. We have

$$f(b) = x*y = f(a)*f(c) = f(a*c).$$

Then $f(b*(a*c)) = 1_Y$ and hence $b*(a*c) \in \ker(f) \subseteq G$. Therefore, $b*(a*c) \in G$. Since $b \in G$, we see that $a*c \in G$, and so $c \in G$. Hence $y = f(c) \in f(G)$. Consequently, $f(G)$ is a pseudo filter of Y. ■

In the following example we show that if f is not surjective in Theorem 14(ii), then we can not implies that $f(G)$ is a pseudo filter of Y.

Example 15. Let Y be pseudo BE-algebra in Example 3(i), and $X = \{1, a, b, c\}$. Define the operations " $*$ " and " $\diamond$ " on X as follows:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	1
b	1	1	1	1
c	1	a	a	1

$\diamond$	1	a	b	c
1	1	a	b	c
a	1	1	c	1
b	1	1	1	1
c	1	a	b	1

Then $(X; *, \diamond, 1)$ is a pseudo BE-algebra. Now, if we consider $f : X \to Y$ as the identity map, then f is a pseudo homomorphism and $f(X) = X$. Can easily seen that $X = \{1, a, b, c\}$ is a trivial pseudo filter of X, but $f(X)$ is not a pseudo filter of Y, because

$$a*d = a \in f(X),\ a \in f(X) \text{ but } d \notin f(X).$$

Corollary 16. *Let $f : X \to Y$ be a pseudo homomorphism. Then*

$$\ker(f) := \{x \in X | f(x) = 1_Y\}$$

is a pseudo filter of X.

Proof. Obviously, $1_X \in \ker(f)$. Hence $\emptyset \neq \ker(f)$. Let $x, x*y \in \ker(f)$. Then $f(x) = f(x*y) = 1_Y$. Therefore, $1_Y = f(x*y) = f(x)*f(y) = 1_Y * f(y) = f(y)$ and $1_Y = f(x \diamond y) = f(x) \diamond f(y) = 1_Y \diamond f(y) = f(y)$. Thus $y \in \ker(f)$. ■

3. Pseudo upper set in BE algebras

Let $x, y \in X$. Define

$$A(x,y) := \{z \in X : x * (y \diamond z) = 1\}.$$

We call $A(x,y)$ an pseudo upper set of x and y.

Note. It is easy to see that, $1, x, y \in A(x,y)$. The set $A(x,y)$, where $x, y \in X$, is not a pseudo filter of X, in general. Also, by using ($pBE4$) and ($pBE5$) we have

$$\begin{aligned} A(x,y) := \{z \in X : x * (y \diamond z) = 1\} &= \{z \in X : y \diamond (x * z) = 1\} \\ &= \{z \in X : y * (x * z) = 1\} \\ &= \{z \in X : y \diamond (x \diamond z) = 1\} \\ &= \{z \in X : y * (x \diamond z) = 1\} = A(y,x). \end{aligned}$$

Example 17. Let X be a pseudo BE-algebra in Example 1(ii). It is easy to check that $A(b,c) = \{1, b, c\}$. Now, we can see that $c*a = b \in A(b,c)$, $c \diamond a = c \in A(b,c)$ and $c \in A(b,c)$, but $a \notin A(b,c)$. Thus $A(b,c)$ is not a pseudo filter of X.

Proposition 18. (i) $A(x,1) \subseteq A(x,y)$,

(ii) *if $A(x,1)$ is a pseudo filter of X and $y \in A(x,1)$, then $A(x,y) \subseteq A(x,1)$.*

(iii) *if there is $y \in X$, such that $y * z = 1$, for all $z \in X$, then $A(x,y) = X = A(y,x)$, for all $x, y \in X$.*

Proof. (i) Let $z \in A(x,1)$. Then $x * (1 \diamond z) = x * z = 1$. Hence $x * (y \diamond z) = y \diamond (x * z) = y \diamond 1 = 1$. Thus $z \in A(x,y)$ and so $A(x,1) \subseteq A(x,y)$.

(ii) Let $A(x,1)$ be a pseudo filter, $y \in A(x,1)$ and $z \in A(x,y)$. Then $x * (y \diamond z) = 1$. Hence $y \diamond z \in A(x,1)$. Now, since $A(x,1)$ is a pseudo filter and $y \in A(x,1)$, $z \in A(x,1)$. Thus $A(x,y) \subseteq A(x,1)$.

(iii) Obviously $A(x,y) \subseteq X$, for all $x \in X$. Let $z \in X$. Then by assumption there is an $y \in X$ such that $y * z = 1$. Hence we have $1 = x \diamond 1 = x \diamond (y * z)$. Therefore, $z \in A(x,y)$. Thus $X \subseteq A(x,y)$ and so $X = A(x,y)$. ∎

Proposition 19. *Let $x \in X$. Then $A(x,1) = \bigcap_{y \in X} A(x,y)$.*

Proof. By Proposition 18(i), we have $A(x,1) \subseteq \bigcap_{y \in X} A(x,y)$. Let $z \in \bigcap_{y \in X} A(x,y)$. Then $z \in A(x,y)$, for all $y \in X$, and so $z \in A(x,1)$. Hence $\bigcap_{y \in X} A(x,y) \subseteq A(x,1)$. ∎

Theorem 20. *Let F be a non-empty subset of X. Then F is a pseudo filter of X if and only if $A(x,y) \subseteq F$, for all $x, y \in F$.*

Proof. Let F be a pseudo filter of X and $x, y \in F$. If $z \in A(x, y)$, then $x*(y \diamond z) = 1 \in F$. Since F is a pseudo filter and $x, y \in F$, by using $(pF2)$, $y \diamond z \in F$ and so by Proposition 5, $z \in F$. Hence $A(x, y) \subseteq F$.

Conversely, suppose that $A(x, y) \subseteq F$, for all $x, y \in F$. Since $x*(y \diamond 1) = x*1 = 1$, we have $1 \in A(x, y) \subseteq F$. Let $a, a*b \in F$. Since $1 = (a*b) \diamond (a*b) = a*((a*b) \diamond b)$, we have $b \in A(a, a*b) \subseteq F$. Hence $b \in F$. Consequently, F is a pseudo filter of X. ■

Theorem 21. *If F is a pseudo filter of X, then*

$$F = \bigcup_{x,y \in F} A(x, y).$$

Proof. Let F be a pseudo filter of X and $z \in F$. Since $z*(1 \diamond z) = z*z = 1$, we have $z \in A(z, 1)$. Hence

$$F \subseteq \bigcup_{z \in F} A(z, 1) \subseteq \bigcup_{x,y \in F} A(x, y)$$

If $z \in \bigcup_{x,y \in F} A(x, y)$, then there exist $a, b \in F$ such that $z \in A(a, b)$. Hence by Theorem 20, we have $z \in F$. This means that $\bigcup_{x,y \in F} A(x, y) \subseteq F$. ■

Theorem 22. *If F is a pseudo filter of X, then*

$$F = \bigcup_{x \in F} A(x, 1).$$

Proof. Let F be a pseudo filter of X and $z \in F$. Since $z*(1 \diamond z) = z*z = 1$, we have $z \in A(z, 1)$. Hence

$$F \subseteq \bigcup_{z \in F} A(z, 1).$$

If $z \in \bigcup_{x \in F} A(x, 1)$, then there exists an $a \in F$ such that $z \in A(a, 1)$, which means that $a*z = a*(1 \diamond z) = 1 \in F$. Since F is a pseudo filter of X and $a \in F$, we see that $z \in F$. This means that $\bigcup_{x \in F} A(x, 1) \subseteq F$. ■

4. Conclusion and future work

In this paper we generalized the notion of BE-algebras and introduced the notion of pseudo BE-algebras, pseudo subalgebras, pseudo filters and pseudo upper set. Furthermore, we prove that every pseudo filter is an union of pseudo upper sets.

We believe that such results are very useful for study in this structure and applied for other algebraic structures. Since quotient algebras are a basic tool for exploring the structures of pseudo BE-algebras and there are close contacts among pseudo filters, pseudo congruence and quotient algebras. In the future work, we are going to consider the notion of the pseudo filters on distributive pseudo BE-algebras to get a quotient algebras induced by this filters. Also, we define pseudo congruence relations on pseudo BE-algebras and give the construction of quotient algebras. We try to define pseudo commutative BE-algebras and other types of pseudo filters in pseudo BE-algebras and discuss on relationship between them.

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