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THE ACTION OF ABELIAN GROUPS ON CERTAIN CONTINUA

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1. In 1939 Kelley [4] proved that if (X, T, Π) is a topological transformation group, where X is a compact connected Hausdorff space and T is a cyclic group, then T leaves invariant a compact connected subset of X which contains no cutpoint of itself. In 1949 Wallace [5] showed that the last result remains true if "T is cyclic" is replaced by "T is abelian". Recently the author, [1], answered a question raised by Wallace by showing that "T is abelian" can be replaced by either of "T is compact" or "T is pointwise regularly almost periodic and X is locally connected."

In 1938 Hamilton [3] proved that if X is a decomposable hereditarily unicoherent compact connected space, and if T is cyclic, then T leaves a proper compact connected subset invariant (Hamilton assumes metrizability, but this assumption can be removed). It seems natural to ask if Hamilton's result can be extended like Kelley's. In this paper we extend Hamilton's result to arbitrary abelian groups. Indeed, we prove that if X is the union of two proper universal subcontinua, and if T is abelian, T leaves a proper compact connected subset of X invariant. This result also includes the result of Wallace [5].

2. A *topological transformation group* is a triple (X, T, Π), where X is a topological space, T is a topological group, and $\Pi\colon TxX \to X$ is a continuous function satisfying $\Pi(t, \Pi(s, x)) = \Pi(ts, x)$ for all $s, t \in T$ and $x \in X$, and $\Pi(e, x) = x$ for all $x \in X$, where e is the identity of T. For convenience of notation we write $\Pi(t, x) = tx$. A non-empty subset of X is invariant under T, or T-invariant, if $TA = A$.

Let X be a compact connected Hausdorff space. X is decomposable if X can be written as the union of two proper closed connected subsets. Otherwise X is indecomposable. X is unicoherent if whenever X is the union of two closed connected subsets A and B, $A \cap B$ is connected. X is hereditarily unicoherent if every closed connected subset of X is unicoherent. If $C \subset X$, we say that C is a *universal subcontinuum* (USC) of X if given a compact connected subset $D \subset X$, $C \cap D$ is compact and connected. The following results are easily verified:

(2.1) *The intersection of arbitrarily many USC is again a USC.*

(2.2) *If $X-x = U \cup V$, where U and V are separated subsets of X, then $U \cup x$ and $V \cup x$ are USC.*

(2.3) *If X is hereditarily unicoherent, every compact connected subset of X is a USC.*

The proof of the following result is to be found in [2]:

(2.4) *If X is a compact connected Hausdorff space and $\varphi: X \to X$ is a homeomorphism, and if no proper compact connected subset of X is invariant under φ, then X is not the union of two proper USC.*

THEOREM. *Let (X, T, Π) be a topological transformation group, where X is a compact connected Hausdorff space which is the union of two proper universal subcontinua, and T is an abelian group. Then there exists a non-empty proper closed connected subset S of X such that $TS = S$.*

Proof. Assume the conclusion of the theorem is false. Then as in [1] we observe:

(2.5) *No proper USC of X contains a T-invariant subset.*

To prove (2.5), assume C is a proper USC of X, $D \subset C$, and D is T-invariant. Let H be the intersection of all USC of X containing D. By (2.1), H is a USC, and $H \subset C$. Furthermore, H is T-invariant: if $t \in T$, $t^{-1}H$ is a USC containing D. Thus $H \subset t^{-1}H$ so that $tH \subset H$. Thus $TH = H$, and the contradiction establishes (2.5).

Now let $X = A \cup B$, where A and B are proper USC of X. We assert that if $t \in T$, then $tA \cap A \neq \emptyset$. For assume there exists $t_0 \in T$ such that $t_0 A \cap A = \emptyset$. Using (2.4) and a theorem of Wallace [5] we deduce that there is a proper non-empty compact connected subset C of X such that $t_0 C = C$, and C is not the union of two proper USC. Let

$$H = \bigcup\{tC; t \in T\}.$$

Since each tC is not the union of two proper USC, it is easy to verify that either $tC \subset A$ or $tC \subset B$. But $t_0 tC = tt_0 C = tC$ and $t_0 A \cap A = \emptyset$ implies $tC \subset B$. This implies that $H \subset B$, and because, as is easy to verify, H is T-invariant, this contradicts (2.5).

By what we have just proved, if $s_1, s_2 \in T$, the equation $X = s_1 A \cup s_1 B$ implies $s_1 A \cap s_2 A \neq \emptyset$, where A and B are as above. Assume for some integer $n \geqslant 2$,

$$\{s_1, s_2, \ldots, s_n\} \subset T \text{ implies } \bigcap\{s_i A; i = 1, \ldots, n\} \neq \emptyset.$$

Let $\{s_1, s_2, \ldots, s_{n+1}\} \subset T$. By (2.1), $s_1 A \cap s_2 A \cap \ldots \cap s_{n-1} A$ is a USC, so that the union of $[s_1 A \cap s_2 A \cap \ldots \cap s_{n-1} A] \cup s_n A$ and $s_{n+1} A$ is a closed connected subset of X, and since $s_{n+1} A$ is a USC, their intersection

$$[s_1 A \cap s_2 A \cap \ldots \cap s_{n-1} A \cap s_{n+1} A] \cup [s_n A \cap s_{n+1} A]$$

is connected. The last set is the union of two closed non-empty subsets, and therefore

$$\bigcap\{s_i A;\, i = 1, \ldots, n+1\} \neq \emptyset.$$

This shows that the collection of closed sets $\{tA;\, t \in T\}$ has the finite intersection property, thus

$$E = \bigcap\{tA;\, t \in T\} \neq \emptyset.$$

It is easily seen that E is T-invariant, and by (2.1), E is a USC of X, hence is compact and connected. Because $E \subset A$, we have a contradiction, and the theorem is proved.

COROLLARY 1. *Let (X, T, Π) be a topological transformation group, where X is a decomposable hereditarily unicoherent compact connected Hausdorff space and T is an abelian group. Then there exists a proper closed connected subset S of X such that $TS = S$.*

Proof. Use (2.3) and the theorem.

COROLLARY 2. *Let (X, T, Π) be a topological transformation group, where T is abelian, and X is a compact connected Hausdorff space. If no proper compact connected subset of X is T-invariant, then X is not the union of two proper USC.*

Proof. By a theorem of [5], there exists a compact connected subset C of X such that (1) C is T-invariant, and (2) no proper compact connected subset of C is T-invariant. By the theorem, C is not the union of two proper USC.

COROLLARY 3 (Wallace [5]). *Let (X, T, Π) satisfy the hypotheses of corollary 2. If no proper compact connected subset of X is T-invariant, then X contains no cut point.*

Proof. Use (2.2) and the theorem.

COROLLARY 4. *Let X be as in the theorem, and T be arbitrary. If no proper closed connected subset of X is left invariant by T, and if $X = A \cup B$, with A and B proper USC of X, then one of A or B is the union of two proper USC.*

Proof. Assume neither A or B is the union of two proper USC. Because $TA \neq A$, there are $t \in T$ such that $tA \not\subset A$. Since tA is not the union of two proper USC, it is easy to see that $tA \subset B$; and since tB is not the union of two proper USC, we have $tB \subset B$ or $tB \subset A$. But $tB \subset B$ would imply $X = B$, and so we must have $tB \subset A$. Then $t(A \cap B) \subset A \cap B \subset A$. It now follows that for every $t \in T$, we have $t(A \cap B) \subset A$, so that $T(A \cap B) \subset A$. Now A is a proper USC of X and $T(A \cap B)$ is T-invariant. This contradicts (2.5), and completes the proof.

It is not known if the theorem of this paper remains true if "T is abelian" is replaced by either of "T is compact" or "T is connected".

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