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ON CONVERGENCE OF INFINITELY DIVISIBLE DISTRIBUTIONS IN A HILBERT SPACE

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Varadhan [3] has generalized the classical formula of Lévy-Khintchine giving the general form of the characteristic functional of an infinitely divisible probability distribution in a Hilbert space.

The aim of this paper is to prove a theorem on convergence of sequences of infinitely divisible probability distributions in a Hilbert space, in terms of the canonical representation.

Let H denote a separable, real Hilbert space. A countably additive, normed measure defined on a field $\mathfrak{B}$ of Borel subsets of H is called a *probability distribution* in H. The convolution of two distributions μ and ν in H is defined by the formula

$$(\mu * \nu)(Z) = \int \mu(Z-h)\nu(dh), \qquad Z \in \mathfrak{B}.$$

The n-th power of a distribution μ in the sense of convolution shall be denoted by μ^{n*} and the convolution of the distributions $\nu_1, \nu_2, \ldots, \nu_n$ by $\prod_{i=1}^{n}{}^{*} \nu_i$. The characteristic functional $\hat{\mu}$ of a distribution μ is defined by the formula

$$\hat{\mu}(h) = \int e^{i(g,h)} \mu(dg).$$

A distribution μ is called *infinitely divisible* if for any natural n there exists a distribution μ_n such that $\mu_n^{n*} = \mu$.

Varadhan ([3], Theorem 5.10) has proved that a functional φ is the characteristic functional of an infinitely divisible distribution in H if and only if it is of the form

$$\varphi(h) = \exp\left[i(x,h) - \frac{1}{2}(Dh,h) + \int \left[e^{i(g,h)} - 1 - \frac{i(g,h)}{1+\|g\|^2}\right] M(dg)\right], \tag{1}$$

where $x \in H$, D is the S-operator (i.e. a non-negative, symmetric operator with a finite trace, see [2]) and M is a σ-finite measure in H which is finite on any complement of the neighbourhood of zero in H, and such that

$$\int_{\|h\|\leqslant 1} \|h\|^2 M(dh) < \infty. \tag{2}$$

The representation (1) is unique. Thus we see that the infinitely divisible distribution μ is determined by three elements: $x \in H$, an S-operator D and the measure M. In this connection we shall write $\mu = [x, D, M]$.

A sequence $\{\mu_n\}$ of distributions in H is said to be *weakly convergent* to a distribution μ ($\mu_n \to \mu$), if for any continuous function bounded in H we have

$$\lim_{n\to\infty} \int f(h)\mu_n(dh) = \int f(h)\mu(dh).$$

A sequence of S-operators $\{D_n\}$ is said to be *compact* if the following conditions are satisfied:

1° $\sup_n \operatorname{tr} D_n < \infty$, where $\operatorname{tr} D$ means the trace of operator D;

2° $\lim_{m\to\infty} \sup_n \sum_{i=m}^{\infty} (D_n e_i, e_i) = 0$, where $\{e_i\}$ is a basis in H.

For $\mu = [x, D, M]$ and $\varepsilon > 0$, put

$$(T^{(\varepsilon)}g, h) = (Dg, h) + \int_{\|u\|\leqslant\varepsilon} (u, g)(u, h)M(du). \tag{3}$$

By (2) the operator $T^{(\varepsilon)}$ described by bilinear form (3) is an S-operator. Now we may formulate the following

THEOREM. *A sequence $\mu_n = [x_n, D_n, M_n]$ of infinitely divisible distributions in H is weakly convergent to the distribution $\mu = [x, D, M]$ if and only if the following conditions are satisfied:*

(i) $x_n \to x$;

(ii) *for any neighbourhood U of zero in H the sequence of measures M_n reduced to $H-U$ is weakly convergent to the measure M (obviously, reduced to $H-U$);*

(iii) *the sequence of operators $\{T_n^{\alpha}\}$ is compact for some $\alpha > 0$;*

(iv) $\lim_{\varepsilon\searrow 0} \overline{\lim_{n\to\infty}} (T_n^{(\varepsilon)}h, h) = \lim_{\varepsilon\searrow 0} \underline{\lim_{n\to\infty}} (T_n^{(\varepsilon)}h, h) = (Dh, h)$ *for every* $h \in H$.

Proof. Necessity. Let $\mu_n = [x_n, D_n, M_n] \to \mu = [x, D, M]$. Hence it follows ([3], Theorem 6.3.) that

(a) the sequence $\{x_n\}$ is compact in H;

(b) for any neighbourhood U of zero in H the sequence of measures $\{M_n\}$ reduced to the set $H-U$ is weakly compact;

(c) for any $\varepsilon > 0$ the sequence of S-operators $\{T_n^{(\varepsilon)}\}$ is compact.

Hence, in particular, we have (iii).

By means of standard reasonings we deduce that from any compact sequence of S-operators $\{Q_n\}$ we can choose a subsequence $\{Q_{k_n}\}$ weakly convergent to some S-operator Q, i.e.

$$(Q_{k_n} g, h) \to (Qg, h) \quad \text{for any } g, h \in H.$$

Let $\{\varepsilon_\nu\}$ be a sequence of positive numbers less than one, monotonically converging to zero. Let $\{k_n\}$ be an increasing sequence of natural numbers such that

(a′) $x_{k_n} \to \bar{x}$;

(b′) for any neighbourhood U of zero in H the sequence of measures $\{M_{k_n}\}$ reduced to $H-U$ is weakly convergent to measure $\bar{M}$ (reduced to $H-U$);

(c′) $\lim\limits_{n\to\infty} (T_{k_n}^{(\varepsilon_\nu)} g, h) = (T^\nu g, h)$ for any $g, h \in H$ and $\nu = 1, 2, \ldots,$ where T^ν $(\nu = 1, 2, \ldots)$ are some S-operators.

Put

$$w_n(\varepsilon, h) = \int\limits_{\|g\|\leqslant\varepsilon} \left[e^{i(g,h)} - 1 - \frac{i(g, h)}{1+\|g\|^2} + \frac{1}{2}(g, h)^2\right] M_n(dg). \tag{4}$$

From compactness of the sequence of operators $\{T_n^{(1)}\}$ it follows that for $\varepsilon < 1$ we have

$$\sup_n \int\limits_{\|g\|\leqslant\varepsilon} \|g\|^2 M_n(dg) \leqslant \sup_n \operatorname{tr} T_n^{(1)} = A < \infty.$$

Hence the estimation

$$\begin{aligned} \sup_n w_n(\varepsilon, h) &\leqslant \int\limits_{\|g\|\leqslant\varepsilon} (\|g\|^3\|h\|^3 + \|h\|\,\|g\|^3) M_n(dg) \\ &\leqslant (\|h\|^3 + \|h\|)\cdot A\cdot\varepsilon \to 0 \quad \text{as} \quad \varepsilon \to 0. \end{aligned} \tag{5}$$

Let us observe, moreover, that the definitions of operators $T_n^{(\varepsilon)}$ imply that the sequence of operators T^ν is monotone, i.e. $(T^\nu h, h) \geqslant (T^{\nu+1} h, h)$ for any $h \in H$. Hence it easily follows that the sequence $\{T^\nu\}$ converges to some S-operator — we denote it by $\bar{D}$.

The logarithm of the characteristic functional of distribution μ_{k_n} can be written in the form

$$\log \hat{\mu}_{k_n}(h) = i(x_{k_h}, h) - \tfrac{1}{2}(T_{k_n}^{(\varepsilon_\nu)} h, h) + w_{k_h}(\varepsilon_\nu, h) + \int\limits_{\|g\|>\varepsilon_\nu} K(g, h) M_{k_n}(dg), \tag{6}$$

where

$$K(g, h) = e^{i(g,h)} - 1 - \frac{i(g, h)}{1+\|g\|^2} \quad \text{and} \quad \nu = 1, 2, \ldots$$

Passing to the limit as $n \to \infty$ we obtain

$$\log \hat{\mu}(h) = i(\bar{x}, h) - \tfrac{1}{2}(T^{\nu} h, h) + w(\varepsilon_\nu, h) + \int_{\|g\| > \varepsilon_\nu} K(g, h) \bar{M}(dg),$$

where $w(\varepsilon_\nu, h) = \lim_n w_{k_n}(\varepsilon_\nu, h)$.

By estimation (5) and the foregoing remarks, passing to the limit as $\nu \to \infty$ we get

$$\log \hat{\mu}(h) = i(\bar{x}, h) - \tfrac{1}{2}(\bar{D}h, h) + \int K(g, h) \bar{M}(dg).$$

In view of the uniqueness of representation (1) we obtain $x = \bar{x}$, $D = \bar{D}$ and $M = \bar{M}$. Since there are no restrictions as to the choice of the sequence $\{k_n\}$, conditions (i) and (ii) follow. Thus only condition (iv) remains to be proved. It is obvious that the limits

$$\lim_{\varepsilon \searrow 0} \overline{\lim_{n \to \infty}} (T_n^{(\varepsilon)} h, h) \quad \text{and} \quad \lim_{\varepsilon \searrow 0} \underline{\lim_{n \to \infty}} (T_n^{(\varepsilon)} h, h)$$

exist for all $h \in H$. It suffices, therefore, to show that

$$\lim_{\varepsilon \searrow 0} \overline{\lim_{n \to \infty}} (T_n^{(\varepsilon)} h, h) \leqslant (Dh, h) \quad \text{and} \quad \lim_{\varepsilon \searrow 0} \underline{\lim_{n \to \infty}} (T_n^{(\varepsilon)} h, h) \geqslant (Dh, h).$$

Suppose that $\lim_{\varepsilon \searrow 0} \overline{\lim_{n \to \infty}} (T_n^{(\varepsilon)} h, h) > (Dh, h)$ for some $h \in H$. Then there exists a sequence of natural numbers $\{l_n\}$ such that

$$\lim_{\varepsilon \searrow 0} \lim_{n \to \infty} (T_{l_n}^{(\varepsilon)} h, h) > (Dh, h),$$

which is impossible, for the preceding considerations imply the existence of a subsequence $\{k_n\}$ of the sequence $\{l_n\}$ and of a sequence $\varepsilon_\nu \searrow 0$ such that

$$\lim_{\nu \to \infty} \lim_{n \to \infty} (T_{k_n}^{(\varepsilon_\nu)} h, h) = (Dh, h).$$

This ends the proof of the necessity of conditions (i)-(iv) of our theorem.

Sufficiency. Conditions (i)-(iii) imply the compactness of the sequence of distributions $\{\mu_n\}$ ([3], Theorem 6.3). By a theorem of Prohorov ([2], § 4) it suffices to show the convergence of the sequence of characteristic functionals $\hat{\mu}_n$ to the functional $\hat{\mu}(h)$ for any $h \in H$. This easily follows from the estimation

$$\begin{aligned} &|\log \hat{\mu}_n(h) - \log \hat{\mu}(h)| \\ &\leqslant |e^{i(x_n, h)} - e^{i(x, h)}| + \Big| \int_{\|g\| \geqslant \varepsilon} |K(g, h) M_n(dg) - \int_{\|g\| \geqslant \varepsilon} K(g, h) M(dg) \Big| + \\ &\quad + |(T_n^{(\varepsilon)} h, h) - (Dh, h)| + \Big| \int_{\|g\| < \varepsilon} K(g, h) M(dg) \Big| + |w_n(h, \varepsilon)|, \end{aligned}$$

where $w_n(h, \varepsilon)$ is defined by formula (4).

Taking $\varepsilon > 0$ sufficiently small, we can make the right-hand side of the inequality, for n sufficiently large, smaller than any $\eta > 0$ given in advance, which ends the proof.

A sequence of distributions $\{\mu_n\}$ shall be called *shift-convergent* if there exists a sequence $\{x_n\}$ of elements of the space H such that the sequence $\{\mu_n * \delta_{x_n}\}$ is weakly convergent (δ_x stands for a one-point distribution concentrated at point x). From the proved theorem, similarly as in the classical case, we can draw conclusions concerning the weak convergence of distributions of the form

$$\mu_n = \prod_{k=1}^{k_n}{}^{*} \mu_{n,k}, \tag{7}$$

where the distributions $\mu_{n,k}$ $(k = 1, 2, \ldots, k_n; n = 1, 2, \ldots)$ are uniformly asymptotically negligible, i.e. for every neighbourhood of zero in H we have

$$\lim_{n\to\infty} \inf_{1\leqslant k\leqslant k_n} \mu_{n,k}(U) = 1.$$

It is known ([3], Theorem 7.6) that sequence (7) is shift-convergent if and only if the sequence of infinitely divisible distributions

$$\lambda_n = \prod_{k=1}^{k_n}{}^{*} e(\theta_{n,k}) \tag{8}$$

is shift-convergent, where

$$e(\nu) = \frac{1}{e}\sum_{k=0}^{\infty}\frac{\nu^{k*}}{k!}, \qquad \theta_{n,k} = \mu_{n,k} * \delta_{-x_{n,k}},$$

$$(x_{n,k}, h) = \int_{\|g\|\leqslant 1} (g, h)\mu_{n,k}(dg) \qquad \text{for any } h \in H.$$

More exactly, the sequence $\{\mu_n * \delta_{x_n}\}$ is convergent if and only if the sequence $\{\lambda_n * \delta_{y_n}\}$ is convergent, where $y_n = x_n + \sum_{k=1}^{k_n} x_{n,k}$.

Our theorem implies the following

COROLLARY. *The sequence* (7), *where the distributions* $\mu_{n,k}$ *are uniformly asymptotically negligible, is shift-convergent if and only if the following conditions are satisfied:*

(a) *for any neighbourhood* U *of zero in* H, *the sequence of measures* $\sum_{k=1}^{k_n} \theta_{n,k}$ *reduced to* $H-U$ *is weakly convergent to some* σ*-finite measure* M *(obviously, reduced to* $H-U$*), finite on* $H-U$ *(for every* U*) and such that*

$$\int_{\|g\|\leqslant 1} \|g\|^2 M(dg) < \infty;$$

(b) $\lim_{\varepsilon\searrow 0}\overline{\lim_{n\to\infty}} = \lim_{\varepsilon\searrow 0}\underline{\lim_{n\to\infty}} \sum_{k=1}^{k_n} \int_{\|g\|\leqslant\varepsilon} (g, h)^2 \theta_{n,k}(dg) = (Dh, h)$, *where* D *is some* S-*operator*;

(c) *the sequence of* S-*operators* $\{B_n\}$ *defined by the bilinear form*

$$(B_n g, h) = \sum_{k=1}^{k_n} \int_{\|u\|\leqslant 1} (u, g)(u, h)\,\theta_{n,k}(du), \qquad n = 1, 2, \ldots,$$

is compact.

Thus the sequence $\{\mu_n\}$ is shift-convergent to the distribution $\mu = [\theta, D, M]$.

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