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ON TEMPERED DISTRIBUTIONS AND BOCHNER-SCHWARTZ THEOREM ON ARBITRARY LOCALLY COMPACT ABELIAN GROUPS

BY

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The aim of this paper is to define the space $\boldsymbol{S}(G)$ of infinitely differentiable rapidly decreasing functions on an arbitrary locally compact Abelian group and by means of this to formulate and prove the Bochner-Schwartz theorem on positive definite distributions. Our definition of the space $\boldsymbol{S}(G)$ is equivalent to the one given by Bruhat [1]. Theorems announced by him, the proofs of which have never been published in details, are formulated and proved in part 3 of the present paper. All these will serve us in proving the Bochner-Schwartz theorem for arbitrary locally compact Abelian groups.

We begin with section 1 which comprises a summary of definitions and facts on Pontriagin duality theory and some structural theorems on locally compact Abelian groups that will be used throughout, introducing at the same time the notation employed. Section 2 is devoted to the definition and some properties of differential operators and polynomials on an arbitrary locally compact Abelian group which lead to the definition of the class of rapidly decreasing infinitely differentiable functions in section 3. Section 4 contains the generalized Bochner-Schwartz theorem and all whatever we have to add to it.

1. Structure theorems on locally compact Abelian groups. All groups here are additively written locally compact Abelian groups. The group of characters of a group G is denoted by Γ, and the value of a $\gamma \in \Gamma$ on $g \in G$ by $\langle g, \gamma \rangle$. Throughout what follows, R will denote the additive group of the reals, Z the additive group of integers and T the multiplicative group of the complex numbers of modulus 1.

For two groups G and H, $G+H$ will denote the direct sum of G and H and G^n will stand for the direct sum of n copies of G.

DEFINITION 1.1. A group of the form $R^k+Z^l+T^m+F$, where F is a finite group, is called an *elementary group*.

Elementary groups are building blocks of an arbitrary group. Partly for completeness and partly to clarify the situation and to introduce a notation convenient for us we include the definitions of the well-known notions of the direct and inverse limit of groups and we recall the Pontriagin duality theory.

DEFINITION 1.2. Let $(\pi, <)$ be a directed system. Suppose that for each $\lambda \in \pi$ a group G_λ is given. We say that a group G is the *direct limit* of groups G_λ, symbolically

$$G = \underset{\rightarrow}{\lim}_\lambda (G_\lambda, h_\lambda),$$

if for each $\lambda \in \pi$ a continuous monomorphism $h_\lambda \colon G_\lambda \to G$ exists, $h_\lambda(G_\lambda)$ being an open subgroup of G, and $\bigcup_\lambda h_\lambda(G_\lambda) = G$ and for each pair $\lambda < \mu$ a continuous monomorphism $h_{\lambda\mu} \colon G_\lambda \to G_\mu$ exists with:

1. $h_{\lambda\lambda}$ is the identity map on G_λ,
2. if $\lambda < \mu$, then $h_\lambda = h_\mu \circ h_{\lambda\mu}$,
3. if $\lambda < \mu < \nu$, then $h_{\lambda\nu} = h_{\mu\nu} \circ h_{\lambda\mu}$.

DEFINITION 1.3. Again let $(\pi, <)$ be a directed system. Suppose that for each $\lambda \in \pi$ a group G^λ is given and for each pair $\lambda < \mu$ with $\lambda, \mu \in \pi$ an epimorphism $x^{\lambda\mu} \colon G^\mu \to G^\lambda$ is given in such a way that

1. $x^{\lambda\lambda}$ is the identity map on G^λ,
2. for any $\lambda < \mu < \nu$ we have $x^{\lambda\nu} = x^{\lambda\mu} \circ x^{\mu\nu}$.

A group G is an *inverse limit* of G^λ, symbolically

$$G = \underset{\leftarrow}{\lim}_\lambda (G^\lambda, x^\lambda),$$

if for each $\lambda \in \pi$ there exists an epimorphism with compact kernel x^λ: $x_\lambda : G \to G^\lambda$ such that $\bigcap_\mu (x^\mu)^{-1}(0) = 0$, and

3. $x^\lambda = x^{\lambda\mu} \circ x^\mu$ for $\lambda < \mu$.

DEFINITION 1.4. Let H be closed subgroup of G. The *annihilator* $H^\perp$ of H is the subgroup of the character group Γ consisting of the homomorphisms which take H into 1.

Let us list some of the properties of $H^\perp$.

PROPOSITION 1.5. (a) $H^{\perp\perp} = H$; (b) $H^\perp$ *is isomorphic to the character group of* G/H; (c) $\Gamma/H^\perp$ *is isomorphic to the character group of* H; (d) *if* H *is compact, then* $H^\perp$ *is open in* Γ (*so* G/H *is discrete*); (e) *if* H *is open in* G, *then* $H^\perp$ *is compact.*

COROLLARIES. *By* (a), *putting* $H = G$, *we see that the character group of* Γ *is* G.

If G *is an inverse limit of groups* (G^λ), *then the character group* Γ *of* G *is a direct limit of the character groups* Γ_λ *of* G^λ.

In fact, the monomorphisms $\eta_{\lambda\mu}\colon \Gamma_\lambda \to \Gamma_\mu$ are given by the formula

$$\langle g^\mu, \eta_{\lambda\mu}\gamma_\lambda\rangle = \langle x^{\lambda\mu} g^\mu, \gamma_\lambda\rangle. \tag{1.1}$$

Conversely, the character group G of a direct limit Γ of groups (Γ_λ) is an inverse limit of the character groups G^λ of Γ_λ, the epimorphisms $x^{\lambda\mu}$ being defined by (1.1) read from the right to the left.

PROPOSITION 1.6. *The character groups of R, Z, T are R, T, Z, respectively. The character group of the direct sum of two groups is the direct sum of the dual groups. Consequently, the character group of an elementary group $R^k+Z^l+T^m+F$ is $R^k+Z^m+T^l+F$.*

DEFINITION 1.7. A group G is said to be *compactly generated*, if there is a compact neighbourhood of zero such that G is generated by the elements of it.

A group is said to be a *Lie group* if it contains an open neighbourhood of zero homeomorphic to a Euclidean space.

PROPOSITION 1.8 (cf. [7]). *The class of compactly generated groups is identical with the class of inverse limits of elementary groups.*

The class of Lie groups is identical with the class of direct limits of elementary groups.

PROPOSITION 1.9 (cf. [7]). *The character group of a Lie group is compactly generated, and vice versa.*

PROPOSITION 1.10 (cf. [7]). *Any group G is*

(I) *a direct limit of compactly generated groups,*

and, on the other hand,

(II) *an inverse limit of Lie groups.*

According to (I) we write

$$G = \lim_{\substack{\lambda\\ \rightarrow}}(G_\lambda, h_\lambda) \quad \text{and} \quad \Gamma = \lim_{\substack{\lambda\\ \rightarrow}}(\Gamma_\lambda, \eta_\lambda).$$

Since h_λ and η_λ are monomorphisms, groups G_λ and Γ_λ may be identified with their images. Thus, for simplification sake, G_λ and Γ_λ will be treated as open subgroups of G and Γ, respectively. Due to this, (II) and the conclusion from Proposition 1.5, groups G and Γ can be presented in the form of inverse limits of groups G^λ and Γ^λ, dual to Γ_λ and G_λ; symbolically,

$$G = \lim_{\substack{\lambda\\ \leftarrow}}(G^\lambda, x^\lambda) \quad \text{and} \quad \Gamma = \lim_{\substack{\lambda\\ \leftarrow}}(\Gamma^\lambda, \chi^\lambda).$$

Then $G^\lambda = G/\Gamma_\lambda^\perp$, $\Gamma^\lambda = \Gamma/G_\lambda^\perp$ and the respective epimorphisms are natural homomorphisms. Kernels of these epimorphisms (otherwise annihilators of Γ_λ and G_λ) will be denoted by $T^\lambda \subset G$ and $\Theta^\lambda \subset \Gamma$, respectively.

Let us assume that λ and μ are such that $T^\mu \subset G_\lambda$. Then $G_\lambda^\mu = G_\lambda/T' = \varkappa^\mu(G_\lambda)$ is an open compactly generated subgroup of the Lie group G^μ, hence, as it is easy to verify, it is an elementary group. Its dual group is also elementary and isomorphic to $\Gamma_\mu^\lambda = \Gamma_\mu/\Theta^\lambda$.

Let us fix μ and let π_μ denote the set of indices for which $T^\mu \subset G_\lambda$. Let us note that

$$G = \underset{\rightarrow}{\lim}_{\lambda\in\pi_\mu}(G_\lambda, h_\lambda) \qquad \text{for any } \mu;$$

thus

$$G^\mu = \underset{\rightarrow}{\lim}_{\lambda\in\pi_\mu}(G_\lambda^\mu, h_\lambda^\mu),$$

where h_λ^μ are injections G_λ^μ in G^μ.

In turn, for a fixed λ let

$$\pi^\lambda = \{\mu\in\pi\colon T^\mu \subset G_\lambda\}.$$

Let $\varkappa_\lambda^\mu$ be the open epimorphism $G_\lambda \to G_\lambda^\mu$ being the restriction of $\varkappa^\mu$ to G_λ: $\varkappa_\lambda^\mu(g) = \varkappa^\mu(g)$, $g\in G_\lambda$.

We observe that $G_\lambda = \underset{\leftarrow}{\lim}_{\nu\in\pi^\lambda}(G_\lambda^\nu, \varkappa_\lambda^\nu)$. Hence the group G can be presented as

$$G = \underset{\rightarrow}{\lim}_\lambda \underset{\leftarrow}{\lim}_\mu G_\lambda^\mu = \underset{\leftarrow}{\lim}_\mu \underset{\rightarrow}{\lim}_\lambda G_\lambda^\mu, \tag{1.2}$$

where indices λ and μ run through the sets for which G_λ^μ is defined.

The group of characters Γ is in turn obtained from elementary groups Γ_μ^λ by a consecutive direct and inverse passage to the limit executed in an arbitrary order,

$$\Gamma = \underset{\rightarrow}{\lim}_\mu \underset{\leftarrow}{\lim}_\lambda \Gamma_\mu^\lambda = \underset{\leftarrow}{\lim}_\lambda \underset{\rightarrow}{\lim}_\mu \Gamma_\mu^\lambda. \tag{1.3}$$

We shall now consider functions on groups. Any function f defined on G_λ^μ can be lifted to the group G in a natural way. In fact, we first write

$$\tilde{f}(g) = f(\varkappa_\lambda^\mu g) \qquad \text{for} \qquad g\in G_\lambda,$$

which as a function defined on the open subgroup G_λ of the group G, is extended to the whole group putting $f \equiv 0$ on $G\setminus G_\lambda$. The obtained function is invariant under translations from T^μ.

A function φ on G constant on the cosets of T^μ determines the function on the elementary group by

$$\varphi_\lambda^\mu(\varkappa_\lambda^\mu g) = \varphi(g) \qquad \text{for} \qquad g\in G_\lambda.$$

If, in addition, the support of φ is contained in G_λ, then $\tilde{\varphi}_\lambda^\mu = \varphi$. The Haar measure on G determines an invariant measure on G_λ^μ, which we denote by $(dg)_\lambda^\mu$. Thus for the integrable function f on G_λ^μ we have

$$\int \tilde{f}dg = \int f(dg)_\lambda^\mu, \tag{1.4}$$

and if φ belongs to $L^1(G)$, then

$$\int \varphi dg = \int \varphi_\lambda^\mu (dg)_\lambda^\mu \text{ if only } \operatorname{supp}\varphi \subset G_\lambda. \tag{1.5}$$

DEFINITION 1.11. If $f \in L^1(G)$, then the function on Γ defined by the formula

$$\hat{f}(\gamma) = \int \langle g, \gamma \rangle f(g)\, dg \tag{1.6}$$

is called the *Fourier transform* of f.

PROPOSITION 1.12 (Plancherel—see e.g. Rudin [8]). (I) *The Fourier transform restricted to $L^1(G) \cap L^2(G)$ is an isometry (with respect to the L^2-norm) onto a dense linear subspace of $L^2(\Gamma)$. Hence it may be extended in a unique manner to an isometry of $L^2(G)$ onto $L^2(\Gamma)$. This extension will be denoted by* $\boldsymbol{F}$.

(II) *If $f \in L^1(G) \cap L^2(G)$ and $\boldsymbol{F}f \in L^1(\Gamma) \cap L^2(\Gamma)$, the inversion formula is valid:*

$$f(g) = \int \langle g, -\gamma \rangle \hat{f}(\gamma)\, d\gamma. \tag{1.7}$$

Let us define the convolution

$$(f * \varphi)(g) = \int f(g-h)\varphi(h)\, dh. \tag{1.8}$$

Note the following properties of the convolution:

LEMMA 1.13 (see e.g. Rudin [8]). *If $f, \varphi \in L^1(G)$, then $f * \varphi \in L^1(G)$ and $\|f * \varphi\|_1 \leqslant \|f\|_1 \|\varphi\|_1$ and also $\widehat{f * \varphi} = \hat{f}\hat{\varphi}$.*

LEMMA 1.14. *If $f = \tilde{f}_\lambda^\mu$ and $f \in L^1(G)$, then $\hat{f} = \tilde{\psi}$, where ψ is a function on Γ_μ^λ.*

Proof. $\hat{f}$ is invariant with respect to the translations by $\delta \in \Theta^\lambda$:

$$\hat{f}(\gamma + \delta) = \int \langle g, \gamma \rangle \langle g, \delta \rangle f(g)\, dg = \int \langle g, \gamma \rangle f(g)\, dg = \hat{f}(\gamma).$$

Still one has to prove that $\operatorname{supp} \hat{f} \subset \Gamma_\mu$. Let $h \in T^\mu$; we have

$$\begin{aligned} \hat{f}(\gamma) &= \int \langle g, \gamma \rangle f(g)\, dg = \int \langle h+g, \gamma \rangle f(h+g)\, dg \\ &= \int \langle h+g, \gamma \rangle f(g)\, dg = \langle h, \gamma \rangle \hat{f}(\gamma). \end{aligned}$$

Hence $(1 - \langle h, \gamma \rangle)\hat{f}(\gamma) = 0$, which means that $\hat{f}(\gamma) = 0$ for $\langle h, \gamma \rangle \neq 1$ which was to be proved.

A function $f(x, n)$, $x \in R^k + T^m$, $n \in Z^l + F$, on the elementary group will be called *differentiable* if for any n the function $f(\cdot, n)$ is differentiable on $R^k + T^m$.

DEFINITION 1.15. We say that $f \in C^k(G)$ if f is a function on G invariant under translations from one of the subgroups T^μ for some μ, and f^μ_λ belongs to $C^k(G^\mu_\lambda)$ for every λ for which G^μ_λ is defined. Infinitely differentiable functions will be denoted by $C^\infty(G)$.

The class of continuous functions will be denoted by $C(G)$, continuous functions with compact supports by $C_0(G)$ and $C_0^\infty(G) = C_0(G) \cap C^\infty(G)$.

PROPOSITION 1.16. *Characters on G are infinitely differentiable.*

Proof. Let $\gamma \in \Gamma$. Then by (1.3)

$$\gamma \in \Gamma_\lambda = \varprojlim_\mu \Gamma^\mu_\lambda,$$

which is the annihilator of T^λ. Thus γ^λ_ν is defined for every ν such that $T^\lambda \subset G_\nu$ and T^λ is the character of the elementary group G^λ_ν. Hence it is infinitely differentiable. By definition of $C^\infty(G)$ the proof follows.

2. Differential operators and polynomials on G. We now define invariant differential operators on the group.

DEFINITION 2.1. The following map is determined on $C^1(G)$ by means of a one-parameter subgroup $a(t), t \in R$, of the group G:

$$C^1(G) \ni \varphi \to D_a\varphi \in C(G),$$

where

$$D_a\varphi(g) = \lim_{t\to 0}\big(\varphi\big(g+a(t)\big)-\varphi(g)\big)t^{-1} \tag{2.1}$$

is called *invariant differential operator of the first order.*

To prove that this definition is correct we have to show that right-hand side of (2.1) exists and belongs to $C(G)$. A homomorphic image of R, the one-parameter subgroup $a(t)$, is contained in a connected component of zero, thus it is contained in each of the open subgroups of group G. Therefore $x^\mu \circ a(t)$ defines a one-parameter subgroup of G^μ_λ, if only G^μ_λ is defined. Since $\varphi \in C^1(G)$, φ^μ_λ exists for some μ. If $g \in G_\lambda$, then

$$D_a\varphi(g) = \lim_{t\to 0}\Big(\varphi^\mu_\lambda\big(x^\mu\big(g+a(t)\big)\big)-\varphi^\mu_\lambda\big(x^\mu(g)\big)\Big)t^{-1}$$

is a continuous function of G, since $\varphi^\mu_\lambda \in C^1(G^\mu_\lambda)$.

The space of differential invariant operators of the first order on G will be denoted by $\boldsymbol{E}_0(G)$; $\boldsymbol{E}(G)$ will denote the algebra generated by $\boldsymbol{E}_0(G)$ by means of addition and superposition, supplemented with the identity map.

PROPOSITION 2.2. *Let f be a function on G constant on cosets of T^μ. Then the following conditions are equivalent*:

(a) $f \in C^\infty(G)$;

(b) $Df \in C(G)$ *for every* $D \in \boldsymbol{E}(G)$.

Proof. (b) results from (a) by the definition of a differential operator.

To prove (b) $\Rightarrow$ (*a*) we have to exhibit a sufficient amount of one-parameter subgroups in group G which mapped down into G^μ_λ by x^μ_λ produce all the differential operators on G^μ_λ. This is done by the following well-known

LEMMA 2.3 (cf. [6]). *Let G be a group, T a compact subgroup, and x the epimorphism from G to G/T. If $a^*(t)$ is a one-parameter subgroup in G/T, there is a one-parameter subgroup $a(t)$ in G such that $x \circ a(t) = a^*(t)$.*

DEFINITION 2.4. Let $D \in \boldsymbol{E}(G)$ and $\gamma \in \Gamma$. Then

$$p\colon \Gamma \ni \gamma \to D\gamma(0)$$

is called a *polynomial* on Γ.

The following proposition, noticed by C. Ryll-Nardzewski, describes the connection between polynomials and homomorphisms of Γ into R:

PROPOSITION 2.5. *There exists a linear isomorphism between $\boldsymbol{E}_0(G)$ and the space of continuous homomorphisms $\Gamma \to R$.*

If D is the element of $\boldsymbol{E}_0(G)$, then the corresp onding polynomial p multiplied by $-i$ is the homomorphism $\Gamma \to R$. Evidently, the map $t \to \langle a(t), \gamma\rangle$ is a character on R, whence it is of the form $\exp(i\lambda t)$. By definition of D, since

$$\langle a(t), \gamma+\delta\rangle = \exp(i\lambda_\gamma t)\exp(i\lambda_\delta t) = \exp\big(i(\lambda_\gamma+\lambda_\delta)t\big),$$

we have $p(\gamma+\delta) = p(\gamma)+p(\delta)$.

To prove that p is a continuous homomorphism we note that if $\gamma_j \in \Gamma$, $j \in I$, tends to γ, then $\langle a(t), \gamma_j\rangle = \exp(i\lambda_j t)$ tends to $\exp(i\lambda_\gamma t) = \langle a(t), \gamma\rangle$ for each t, which shows that λ_j tends to λ.

Conversely, a continuous homomorphism $\Gamma \to R$ determines, by the duality theorem, a one-parameter subgroup by the formula

$$\langle a(t), \gamma\rangle = \exp\big(itp(\gamma)\big)$$

to which in turn there corresponds the polynomial $p(\gamma)$.

PROPOSITION 2.6. *If p^* is a linear polynomial on Γ^μ_λ, then there exists a polynomial p on Γ such that $p^\mu_\lambda = p^*$.*

Proof. It is the dual theorem to Lemma 2.4. In fact, let D^λ_μ be a differential operator on G^λ_μ induced by p^*. Then, by Lemma 2.4, there exists a differential operator D on G, the restriction of which to $C^\infty(G^\lambda_\mu)$ equals D^λ_μ. The polynomial on Γ corresponding to D is the required polynomial p.

3. Class $S(G)$. The class $S(G)$ of infinitely differentiable rapidly decreasing functions on arbitrary locally compact Abelian groups was defined by Bruhat in [1], where most of the properties of $S(G)$ analogous to the corresponding ones for the ordinary class $S(R^k)$ are listed without proofs. Here we prove two of the fundamental properties of $S(G)$, namely that the Fourier transform maps isomorphically $S(G)$ onto $S(\Gamma)$ and that $S(G)$ is dense in $L^1(G)$.

DEFINITION 3.1. We say that f *belongs to* $S(G)$ if

(a) $f \in C^\infty(G)$,

(b) $\tilde{f}^\mu_\lambda = f$ for some λ and μ, and

(c) for every polynomial p and every differential operator D the product pDf belongs to $L^2(G)$.

In (c) the condition $pDf \in L^2(G)$ could be equivalently replaced by $pDf \in L^p(G)$ for any $1 \leqslant a \leqslant \infty$.

In the set $S^\mu_\lambda(G)$ consisting of functions from $S(G)$ invariant under translations from T^μ whose support is contained in G_λ the topology is introduced by means of the set of semi-norms

$$\|f\|_{p,D} = \|pDf\|_2, \quad \text{where} \quad p \in P(G) \text{ and } D \in E(G)$$

($P(G)$ denotes the space of polynomials on G).

The topology in $S(G)$ is then defined as the strongest topology for which the injections $S^\mu_\lambda(G) \to S(G)$ are continuous.

THEOREM 3.2. *The Fourier transform is a linear continuous homomorphism from* $S(G)$ *onto* $S(\Gamma)$.

Proof. First we note that differentiation and multiplication by a polynomial map $S(G)$ into $S(G)$. The first fact follows from definition of $S(G)$, the second is the consequence of the inclusion $P(G) \subset C^\infty(G)$, and the fact that $D: P(G) \to P(G)$ and $D(pf) = (Dp)f + pDf$.

(1) We now prove that a function f of the class $S(G)$ is dg-integrable) Let $f = \tilde{\varphi}$, where $\varphi \in S(G^\mu_\lambda)$. Let $\overline{p} \in P(G_\lambda)$ have the popertyr $|\overline{p}|^{-1} \in L^2(G^\mu_\lambda.$ and let p be an extension of $\overline{p}$ onto G (cf. Proposition 2.6). Then $(f)^\mu_\lambda = f^\mu_\lambda |\overline{p}|\, |\overline{p}|^{-1}$ is $(dg)^\mu_\lambda$-integrable as the product of two functions from $L^2(G^\mu_\lambda)$. Hence also $fp \in L^1(G)$.

(2) Thus the Fourier transform is well defined on $S(G)$ and by Plancherel theorem and Lemma 1.14 the Fourier transform of $f \in S(G)$ belongs to $L^2(\Gamma)$, it is constant on the cosets of a Θ^λ, and the support of it is in Γ_μ.

Moreover, $\hat{f}$ is differentiable and

$$D\hat{f} = \widehat{pf}, \tag{3.1}$$

where $p \in \boldsymbol{P}(G)$ is the polynomial generated by $D \in \boldsymbol{E}(\Gamma)$. In fact, let $D \in \boldsymbol{E}_0(\Gamma)$. Then

$$D\hat{f}(\gamma) = \lim_{t\to 0} \int \big(\langle g, a(t)\rangle - 1\big) t^{-1} \langle g, \gamma\rangle f(g)\, dg.$$

For any t the modulus of the integrand is dominated by pf. Applying Lebesgue's theorem, we obtain

$$\begin{aligned} D\hat{f}(\gamma) &= \int \lim_{t\to 0} \big(\exp\big(tp(g)\big) - 1\big) t^{-1} \langle g, \gamma\rangle f(g)\, dg \\ &= \int p(g) \langle g, \gamma\rangle f(g)\, dg = \widehat{pf}(\gamma). \end{aligned}$$

This formula can be extended on $D \in \boldsymbol{E}(\Gamma)$ by induction. But the Fourier transform of an integrable function is continuous on Γ, which proves the differentiability of $\hat{f}$ (Proposition 2.2). Another consequence of the formula above is that differentiation maps $\boldsymbol{F}(\boldsymbol{S}(G))$ into itself.

(3) We have

$$\widehat{Df}(\gamma) = (-1)^{|a|} p(\gamma) \hat{f}(\gamma), \tag{3.2}$$

where $|a|$ is the order of D.

Assuming that $D \in \boldsymbol{E}_0(G)$, we have

$$\widehat{Df}(\gamma) = \int \lim_{t\to 0} \big(f\big(g + a(t)\big) - f(g)\big) t^{-1} \langle g, \gamma\rangle\, dg.$$

The quotient $\big(f\big(g+a(t)\big) - f(g)\big) t^{-1}$ tends to Df in the $L^1(G)$-norm. We have

$$\begin{aligned} \Big\|\big(f\big(g+a(t)\big) - f(g)\big) t^{-1} - Df\Big\|_1 &= t^{-1} \Big\| \int_0^t Df\big(g + a(\tau)\big) - Df(g)\, d\tau \Big\|_1 \\ &\leqslant t^{-1} \int_0^t \big\| Df\big(g + a(\tau)\big) - Df(g) \big\|_1 d\tau. \end{aligned}$$

Since the translations act continuously in $L^1(G)$, we obtain the desired result. Thus

$$\widehat{Df}(\gamma) = \lim_{t\to 0} \Big(\int f\big(g + a(t)\big) t^{-1} \langle g, \gamma\rangle\, dg - \int t^{-1} f(g) \langle g, \gamma\rangle\, dg \Big)$$

and

$$\begin{aligned} \widehat{Df}(\gamma) &= \lim_{t\to 0} \int f(g) \big(\langle a(t), -\gamma\rangle - 1\big) t^{-1} \langle g, \gamma\rangle\, dg \\ &= \lim_{t\to 0} t^{-1} \big(\langle a(t), -\gamma\rangle - 1\big) \int \langle g, \gamma\rangle f(g)\, dg = -p(\gamma) \hat{f}(\gamma). \end{aligned}$$

Formula (3.2) follows now by induction.

As in point (1), it follows from formula (3.2) that

$$p(\gamma)\hat{f}(\gamma) \in L^1(\Gamma) \cap L^2(\Gamma)$$

for every polynomial from $\boldsymbol{P}(\Gamma)$. In virtue of Proposition 1.12 (II) we have

$$f(-\gamma) = \int \langle g, \gamma \rangle \hat{f}(g)\, dg,$$

whence

$$\boldsymbol{F}\big(\boldsymbol{S}(G)\big) \supset \boldsymbol{S}(\Gamma). \tag{3.3}$$

By (1) and (2) we obtain

$$\boldsymbol{F}\big(\boldsymbol{S}(G)\big) \subset \boldsymbol{S}(\Gamma), \tag{3.4}$$

and so $\boldsymbol{F}\big(\boldsymbol{S}(G)\big) = \boldsymbol{S}(\Gamma)$.

(4) It remains to prove that the Fourier transform is a continuous map on $\boldsymbol{S}(G)$. Let $D \in \boldsymbol{E}(\Gamma)$, $p \in \boldsymbol{P}_0(\Gamma)$, $\|\hat{f}\|_{p,D} = \|pD\hat{f}\|_2 = \|D_1 p_1 f\|_2$ for some $D_1 \in \boldsymbol{E}_0(G)$, and $p_1 \in \boldsymbol{P}(G)$. This, by (2) and (3) and Proposition 1.12 (I), yields

$$D_1 p_1 f = (D_1 p_1) f + p_1 D f;$$

but $D_1 p_1 = p_2 \in \boldsymbol{P}(G)$ and so

$$\|D_1 p_1 f\|_2 \leqslant \|f\|_{p_1, I} + \|f\|_{p_1, D_1}.$$

THEOREM 3.3. *$\boldsymbol{S}(G)$ is dense in $L^1(G)$.*

Proof. Since $\boldsymbol{S}(G_\lambda^\mu)$ is evidently dense in $L^1(G_\lambda^\mu)$, it remains to prove that $\bigcup_{\mu,\lambda} \tilde{L}(G_\mu^\lambda)$ is dense in $L^1(G)$. Let us define a family of projectors by

$$E^\mu f(g) = |T^\mu|^{-1} \int f(g+k)\, dk^\mu,$$

where dk^μ is the Haar measure on T^μ, and $|T^\mu|$ denotes $\int dk^\mu$. Then

$$\begin{aligned} \|E^\mu f - f\|_1 &= |T^\mu|^{-1} \Big\| \int f(g+k) - f(g)\, dk^\mu \Big\|_1 \\ &\leqslant |T^\mu|^{-1} \int \|f(g+k) - f(g)\|_1\, dk^\mu \leqslant \sup_{k \in T^\mu} \|f(g+k) - f(g)\|_1. \end{aligned}$$

The above inequality implies that $\chi_\lambda E^\mu f$ is a Moore-Smith sequence of functions from $L^1(G)$ convergent to f in $L^1(G)$-norm. This completes the proof.

4. Bochner-Schwartz theorem. Now we formulate and prove the Bochner-Schwartz theorem for an arbitrary locally compact Abelian group.

In the finite-dimensional space of linear polynomials on the elementary group Γ_μ^λ we choose a basis consisting of polynomials $\{p_i\}_1^r$.

Then the polynomials

$$p^{a} = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}, \quad \text{where} \quad a = (a_1, \dots, a_r),$$

form a basis of $P(\Gamma_\mu^\lambda)$.

Let $|a| = a_1 + \dots + a_r$ and let the differential operators $\{D_i\}_1^r$ be those which generate $\{p_i\}_1^r$.

A topology in $C_0^\infty(G)$ is defined as follows. In the subspace $\boldsymbol{D}^\mu(K)$, consisting of functions with supports in a compact set $K \subset G$ and invariant under translations from T^μ, the topology is given by the set of semi-norms

$$\|f\|_p' = \Big(\sum_{|a| \leqslant p} \|D^a f\|_2^2\Big)^{1/2}. \tag{4.1}$$

In the whole space $\boldsymbol{D}(G)$ we define the topology as the strongest one for which the injections $D^\mu(K) \to D(K)$ are continuous. Clearly, $\boldsymbol{D}(G) \subset \boldsymbol{S}(G)$ and the topology of $\boldsymbol{D}(G)$ is stronger.

DEFINITION 4.1. The elements of the space $\boldsymbol{D}'(G)$ of continuous linear functionals on $\boldsymbol{D}(G)$ will be called *distributions* on G. Respectively, $\boldsymbol{S}'(G)$ denotes the space of tempered distributions, i.e. the space of linear continuous functionals on $\boldsymbol{S}(G)$.

We shall say that T is a *positive-definite distribution* if

$$T(\varphi^* * \varphi) \geqslant 0$$

for every $\varphi \in \boldsymbol{D}(G)$, where $\varphi^* * \varphi(g) = \int \varphi(g+h)\bar{\varphi}(h)\,dh$.

The following theorem, proved by Maurin [4] and [5], generalizes the classical Bochner-Schwartz theorem:

THEOREM 4.2. *Let G be a separable group. Every positive-definite element $T \in \boldsymbol{D}'(G)$ is represented by a positive Borel measure m as follows:*

$$T(\varphi) = \int \hat{\varphi}(\gamma)\,dm(\gamma). \tag{4.2}$$

REMARK. The assumption of separability can be removed.

For every μ and λ the functional T_λ^μ defined by the formula $T_\lambda^\mu(\varphi) = T(\tilde{\varphi})$, $\varphi \in D(G_\lambda^\mu)$ is a positive-definite distribution on G_λ^μ. We have

$$T(\tilde{\varphi}) = T_\lambda^\mu(\varphi) = \int \hat{\varphi}\,dm_\mu^\lambda = m(\tilde{\varphi}). \tag{4.3}$$

In this way we obtain a positive functional m defined on those functions of $C_0(\Gamma)$ which are constant on cosets of certain Θ^λ. However, it is a functional continuous in the compact convergence in $C_0(\Gamma)$. Namely, let ψ be a function from $C_0(\Gamma)$ equal to 1 on a compact set K. Now, if φ is a function from $C_0^\infty(\Gamma)$ with a support in K and constant on the cosets of Θ^λ, then

$$|m(\varphi)| = |m(\varphi\psi)| \leqslant \int |\varphi|\,|\psi|\,dm_\mu^\lambda \leqslant \sup|\varphi| \int |\psi|\,dm_\mu^\lambda = \sup|\varphi|\, m(|\psi|).$$

Thus the functional $m(\cdot)$ can be extended to a positive functional on $C_0(\Gamma)$, continuous in the compact convergence, which is a measure on Γ.

THEOREM 4.3. *The measure m can be extended in a unique way to a tempered distribution on Γ.*

The approach to the proof which follows has been suggested by A. Hulanicki.

In case $G = R^k$, the classical Bochner-Schwartz theorem states that the measure m which is defined by a positive-definite distribution on R^k by means of formula (4.2) is of polynomial growth. This means that there exists a polynomial p on R^k such that $|p|^{-1}$ belongs to $L^1(R^k, m)$.

We need here a more detailed formulation of the thesis.

Let $\|\cdot\|_q$ be a semi-norm such that the restriction of T to $\boldsymbol{D}(K)$ is bounded with respect to $\|\cdot\|_q$. Hence

$$|T(\varphi)| = \left|\int \hat{\varphi}\, dm\right| \leqslant A \sum_{|a|\leqslant q} \|D^a \varphi\|_2 = A \sum_{|a|\leqslant q} \|p^a \hat{\varphi}\|_2, \tag{4.4}$$

where p^a is the polynomial generated by D^a.

It is known (cf. [3], p. 200-206) that there is a q for which (4.4) holds for all compact subsets K in R^k and that then the formula

$$\int |p|^{-1} dm \leqslant A \sum_{|a|\leqslant q} \|p^a p^{-1}\|_2 \tag{4.5}$$

is valid.

To prove Theorem 4.3 we note that a straight-forward application of the classical Bochner-Schwartz theorem gives the result for elementary groups. In fact, let $G = R^k + T^l + Z^m + F$ be an elementary group and let T be a positive-definite distribution on $\boldsymbol{D}(G)$. Then, by Maurin's theorem, there exists a measure on $\Gamma = R^k + T^m + T^l + F$ such that

$$T(\varphi) = \int \hat{\varphi}\, dm \quad \text{for all } \varphi \in \boldsymbol{D}(G). \tag{4.6}$$

We shall prove that there exist polynomials r^a and z^β on R^k and Z^l, respectively, such that

$$\int (1 + |r^a|)(1 + |z^\beta|)\, dm < \infty. \tag{4.7}$$

Let $\varphi_n(r, t, z, f) = \psi(r)\chi_n(t)\delta(z)\delta(f)$ be a function on G, where $\psi \in D(R^k)$ and $\chi_n(t) = \exp(i\langle n, t\rangle)$ for $n \in Z^l$ and $t \in T^l$. The formula

$$\boldsymbol{D}(R^k) \ni \psi \to T_n(\psi) = T(\varphi_n) = \int \hat{\psi}(r)\hat{\chi}(z)\hat{\delta}(t)\hat{\delta}(f)\, dm$$

defines a family of positive-definite distributions on R^k.

Let K be a compact subset of G which is a direct sum of K_1 and K_2, where $K_1 \subset R^k$ and $K_2 \subset T^l+Z^m+F$. There exists a seminorm $\|\cdot\|_q$ such that for $\varphi_n \in \boldsymbol{D}(G)$ with supports in K we have

$$|T(\varphi_n)| \leqslant A\,\|\varphi_n\|_q.$$

Hence

$$|T_n(\psi)| = |T(\varphi_n)| \leqslant A \sum_{|a|\leqslant q} \|D^a\varphi_n\|_2 = A \sum_{|a|\leqslant q} \|p^a\hat{\varphi}_n\|_2. \tag{4.8}$$

Let us choose $p_1(r)$ and $p_2(z)$ in such a way that

$$\sum_{|a|\leqslant q} |p^a|^2(r,z) \leqslant |p_1(r)p_2(z)|.$$

Since $\hat{\chi}_n$ are characteristic functions of one-point sets $\{n\}$, $n \in Z^l$, we obtain

$$|T_n(\psi)| \leqslant |p_1(n)|\,B\,\|p_2\hat{\psi}\|_2 \quad \text{for} \quad \psi \in \boldsymbol{D}(K) \subset \boldsymbol{D}(R^k). \tag{4.9}$$

By (4.5), there exists a polynomial $p(r) = 1+|r^a|$ such that

$$\int (1+|r^a|)^{-1}\hat{\chi}_n(z)\,\hat{\delta}(t)\,\hat{\delta}(f)\,dm \leqslant |p_1(n)|\,B\,\|p_2(1+|r^a|)\|_2. \tag{4.10}$$

Now, we can assert that (4.7) is valid provided β is chosen in such a way that

$$\sum_{n\in Z^l} p_1(n)(1+|n^\beta|) < \infty.$$

Returning to the case of an arbitrary group, we have the family of positive measures on elementary groups defined by

$$\int \varphi\,(dm)_\mu^\lambda = \int \tilde{\varphi}\,dm \quad \text{for} \quad \varphi \in \boldsymbol{D}(\Gamma_\mu^\lambda), \tag{4.11}$$

and the corresponding set of positive-definite distributions by

$$T_\lambda^\mu(\psi) = \int \hat{\psi}\,(dm)_\mu^\lambda. \tag{4.12}$$

In virtue of the theorem proved for elementary groups, $(dm)_\mu^\lambda$ can be continuously extended to a tempered distribution on Γ, which means that equality (4.11) read in the opposite direction gives the extension of measure m to $\boldsymbol{S}(\Gamma)$.

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