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ON A PROBLEM OF E. MICHAEL CONCERNING TOPOLOGICAL DIVISORS OF ZERO

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The aim of the present note is to give an answer to a problem posed by Michael in [2] concerning the equivalence of two notions generalizing the concept of topological divisors of zero in Banach algebras. The answer is negative and a counter-example is presented in the Theorem of this note.

The following terminology will be used in the sequel.

A *topological algebra* is an algebra (a linear ring) A over the real or complex scalars with a Hausdorff topology making the linear space A a topological linear space and the operation of multiplication separately continuous. A topological algebra with a unit is called a *Q-algebra* provided the set of invertible elements is open.

A topological algebra is said to be *complete* if it is complete when considered as a topological linear space, i.e. if for any directed set $\mathfrak{A}$ and for any sequence $(x_a)_{a\in\mathfrak{A}} \subset A$ the condition $\lim\limits_{a,\beta\in\mathfrak{A}} (x_a - x_\beta) = 0$ implies the convergence of the sequence (x_a).

Let L_x and R_x denote the linear operators $A \to A$ of left- and right-side multiplication by x: $L_x y \overset{\text{df}}{=} xy$, $R_x y \overset{\text{df}}{=} yx$. An element $x \in A$ is called a *left* (*right*) *topological divisor of zero* (short, *t.d.z.*) provided $x \neq 0$ and the operator L_x (resp. R_x) is not an isomorphism, i.e. a linear homeomorphism of A onto xA (Ax).

If A is metric, an alternative description of t.d.z.'s is this:

An element $x \in A$ is a t.d.z. if and only if $x \neq 0$ and there exists a sequence (x_n) of elements of A with $x_n \not\to 0$ and $xx_n \to 0$ (resp. $x_n x \to 0$). (In the non-metric case this description remains valid with the only exception that the sequence need not be countable and should be replaced by one-indexed directed set).

It follows that an element which is a t.d.z. cannot be invertible.

The class of Banach algebras (short, B-algebras) is contained in the class of topological algebras (in fact in the class of complete Q-algebras). A wider subclass of topological algebras is that formed by all m-convex algebras — a notion whose definition reads as follows:

A topological algebra A is called *locally multiplicatively convex* (short: *m-convex*) provided its topology is locally convex and may be given by means of a system of submultiplicative pseudonorms, i.e. a system $(\|\cdot\|_\alpha)$ of pseudonorms satisfying the condition

$$\|xy\|_\alpha \leqslant \|x\|_\alpha \|y\|_\alpha \quad \text{for} \quad x, y \in A.$$

The notion of an m-convex algebra was first introduced by Arens [1] and examined in detail by Michael in [2].

Some facts from the theory of B-algebras are valid also for m-convex algebras. Most of the principal results are obtained in [2] by means of the following argument:

Suppose that the topology of an m-convex algebra A is given by means of a system $(\|\cdot\|_\alpha)_{\alpha\in\mathfrak{A}}$ of submultiplicative pseudonorms satisfying the additional condition

(m) together with any two pseudonorms $\|\cdot\|'$, $\|\cdot\|''$, the pseudonorm $\max(\|\cdot\|', \|\cdot\|'')$ is in the system

(an unessential restriction, since any system of pseudonorms may be extended to an equivalent one satisfying (m)). The relation $\alpha \leqslant \beta$ defined for $\alpha, \beta \in \mathfrak{A}$ by $\|\cdot\|_\alpha \leqslant \|\cdot\|_\beta$ directs then the set $\mathfrak{A}$. Let A_α denote the quotient algebra A/N_α, where $N_\alpha = \{x \in A : \|x\|_\alpha = 0\}$; the pseudonorm $\|\cdot\|_\alpha$ induces a norm in A_α; denote by $\bar{A}_\alpha$ the completion of A_α in this norm. For any two indices $\alpha, \beta \in \mathfrak{A}$ with $\alpha \leqslant \beta$ there exists a unique continuous homomorphism $\pi_{\alpha\beta}\colon \bar{A}_\beta \to \bar{A}_\alpha$ which makes the diagram

$$\begin{array}{ccc} & A & \\ \pi_\alpha \swarrow & & \searrow \pi_\beta \\ \bar{A}_\alpha & \xleftarrow[\pi_{\alpha\beta}]{} & \bar{A}_\beta \end{array}$$

commute — here π_α and π_β denote the natural projections $A \to \bar{A}_\alpha$ and $A \to \bar{A}_\beta$. The B-algebras $\bar{A}_\alpha$ together with the homomorphisms $\pi_{\alpha\beta}$ form an inverse sequence; the inverse limit of this sequence is isomorphic to the algebra A if the latter is complete. Proofs of these facts may be found in [2].

In order to obtain some results analoguous to those concerning the properties of t.d.z.'s in B-algebras Michael introduces in [2] a weaker concept of an m-t.d.z., using the facts just mentioned about the structure of m-convex algebras.

An element $x \neq 0$ of an m-convex algebra A will be called an *m-left* (*right*) *t.d.z.* provided for any system of submultiplicative pseudonorms giving the topology of A and satisfying condition (m) an index $\alpha \in \mathfrak{A}$ exists such that $\pi_\alpha x$ is a left (right) t.d.z. in $\bar{A}_\alpha$.

Michael uses a slightly different terminology, namely, he calls m-t.d.z.'s just t.d.z.'s and what is usually called a t.d.z., is called by Michael a *strong* t.d.z.

An element which is a t.d.z. is necessarily an m-t.d.z. Michael does not know whether the two concepts are actually equivalent.

The following facts are well known in the theory of B-algebras (see for instance Rickart [3]):

Let A be a B-algebra with a unit element e and let $x \in A$ and $x \neq 0$. Then

(i) if x is in the closure of the set of all invertible elements but is itself not invertible, then x is a two-sided t.d.z.;

(ii) if $\lambda \in \mathrm{bdry}\, \sigma(x)$, where $\sigma(x)$ denotes the spectrum of x, then the element $x - \lambda e$ is a two-sided t.d.z. Consequently, the radical of A consists entirely of two-sided t.d.z.'s (and, of course, the zero element);

(iii) if A has no t.d.z.'s, then A is either the field of the reals or of the complexes or the division algebra of real quaternions.

Michael shows in [2] that (i) and (ii) remain valid if the supposition that A is a B-algebra is replaced by one stating that A is a complete m-convex algebra and if m-t.d.z. is always written instead of t.d.z. (iii) remains true under the additional supposition that A be a complete m-convex Q-algebra.

The supposition that the notion of m-t.d.z. is equivalent to that of t.d.z. is contradicted by the simple example given below. It shows that neither of the facts (i)-(iii) remains true under the reformulation consisting in the sole replacement of B-algebras by complete m-convex Q-algebras. This example is given in the proof of the

THEOREM. *There exists a topological algebra X with the following properties*:

(1) *X is a commutative metric complete m-convex Q-algebra with a unit element.*

(2) *The set G of all invertible elements of X is dense in X.*

(3) *The radical of $X = X \setminus G = \mathrm{bdry}\, G \neq \{0\}$.*

(4) *X has no topological divisors of zero.*

Proof. (1) X is defined as the algebra of all real- or complex-valued sequences $(\xi_n)_{n=0}^{\infty}$ with convolution multiplication and with the topology of coordinatewise convergence. This topology may be given by means of pseudonorms $\|\cdot\|_n$ defined by

$$\|x\|_n \stackrel{\mathrm{df}}{=} |\xi_0| + \ldots + |\xi_n|$$

for

$$x = (\xi_0, \xi_1, \xi_2, \ldots), \qquad n = 0, 1, 2, \ldots$$

These pseudonorms are submultiplicative, since for $x = (\xi_n)_{n=0}^{\infty}$, $y = (\eta_n)_{n=0}^{\infty}$ we have

$$xy = (\zeta_n)_{n=0}^{\infty}, \quad \text{where} \quad \zeta_n = \xi_0\eta_n+\ldots+\xi_n\eta_0$$

and

$$\begin{aligned} \|xy\|_n &= |\zeta_0|+\ldots+|\zeta_n| \\ &= |\xi_0\eta_0|+|\xi_0\eta_1+\xi_1\eta_0|+\ldots+|\xi_0\eta_n+\ldots+\xi_n\eta_0| \\ &\leqslant (|\xi_0|+\ldots+|\xi_n|)(|\eta_0|+\ldots+|\eta_n|) \\ &= \|x\|_n\|y\|_n. \end{aligned}$$

The algebra X is countably normed and, consequently, metric. It is also complete in view of the fact that its topology is that of coordinatewise convergence. Thus a sequence of its elements is Cauchy if and only if it is coordinatewise Cauchy and the completeness follows from the completeness of the field of scalars.

The unit element of the algebra X is the element $e = (1, 0, 0, \ldots)$. Direct computation shows that the equation $xy = e$, where $x = (\xi_0, \xi_1, \xi_2, \ldots)$, has a solution in y if and only if $\xi_0 \neq 0$. Elements x for which this holds form an open set and, consequently, X is a Q-algebra.

(2) G is dense in X, since if $x \in X \setminus G$, then $x = (0, \xi_1, \xi_2, \ldots)$, the elements

$$x_n = x+\frac{1}{n}e = \left(\frac{1}{n}, \xi_1, \xi_2, \ldots\right)$$

are in G and $x = \lim x_n$.

(3) Write $z = (0, 1, 0, 0, \ldots)$. For any $x = (\xi_n)_{n=0}^{\infty} \in X$ we have

$$x = \sum_{n=0}^{\infty} \xi_n z^n,$$

so the algebra X is generated by z. Thus every continuous functional f is determined by its value on z:

$$f(x) = \sum_{n=0}^{\infty} \xi_n f(z)^n \quad \text{for} \quad x = (\xi_n) \in X.$$

Since the algebra contains all the sequences $x = (\xi_n)$, this series may be convergent for all $x \in X$ only in the case when $f(z) = 0$. It follows that there is only one continuous functional f:

$$f(x) = \xi_0 \quad \text{for} \quad x = (\xi_n) \in X.$$

Therefore

$$\text{radical of } X = \{x \in X : f(x) = 0\} = \{x = (\xi_n) \in X : \xi_0 = 0\}$$
$$= X \setminus G \neq \{0\}.$$

(4) For $x = (\xi_0, \xi_1, \xi_2, \ldots)$ we have $zx = (0, \xi_0, \xi_1, \ldots)$. The topology is that of coordinatewise convergence and thus a sequence $x_1, x_2, \ldots$ of elements of X converges to zero if and only if $zx_n \to 0$. An element $x \in X$ is thus a t.d.z. if and only if so is zx, and, more generally, — by induction — if and only if $z^k x$ is a t.d.z. (k — arbitrary posivite integer). But any non-zero element of A may be written in the form $x = z^k x_0$, where $x_0 \in G$; in fact, if $x = (\underbrace{0, \ldots 0}_{k \text{ times}}, \xi_0, \xi_1, \ldots)$ with $\xi_0 \neq 0$, then $x = z^k x_0$, where $x_0 = (\xi_0, \xi_1, \ldots) \in G$. The existence of a t.d.z. in X would then imply the existence of an invertible t.d.z., which is impossible.

This contradiction concludes the proof.

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