ON THE PROXIMATE FIXED-POINT PROPERTY
FOR MULTIFUNCTIONS

BY

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In this paper* a study of ε-continuity is initiated from the viewpoint of multifunctions, and a proximate fixed-point property is developed for various classes of ε-continuous multifunctions. The purpose of this paper is to generalize some fixed-point theorems of Klee [2] and Yandl [7] to the setting of multifunctions. The main results are that if a metric space $X$ has the proximate fixed-point property for multifunctions, then so does every compact $m$-retract of $X$ and that if, further, $X$ is compact, then every metric homeomorph of $X$ has the proximate fixed-point property for multifunctions. The author and R. E. Smithson have proved in related papers [4], [5] that non-empty, compact, convex subspaces of locally convex, Hausdorff linear topological spaces and trees both have the proximate fixed-point property for various classes of multifunctions.

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A multifunction $F$ on a space $X$ into a space $Y$ is a correspondence between elements of $X$ and non-empty subsets of $Y$. To be precise $F \subset X \times Y$, and, for each $x \in X$, $\pi_2((\{x\} \times Y) \cap F) \neq \emptyset$ where $\pi_2$ is the second projection on $X \times Y$ into $Y$ and $\emptyset$ denotes the empty set. In particular, every (single-valued) function is a multifunction. We shall write $F : X \rightarrow Y$ for a multifunction on $X$ into $Y$ and $F(x)$ for the set $\pi_2((\{x\} \times Y) \cap F)$. If $A \subset X$, then $F(A) = \bigcup \{F(x) : x \in A\}$. If $U$ is a subset of some topological space $X$, then $U^o$ and $U^*$ denote the interior and closure, respectively, of $U$ in $X$.

DEFINITION. Let $F : X \rightarrow Y$. Then $F$ is lower semi-continuous (l.s.c.) if and only if for each $x \in X$ and for each $V = V^o \subset Y$ such that $F(x) \cap V \neq \emptyset$ there exists $U = U^o \subset X$ with $x \in U$ such that $F(x') \cap V \neq \emptyset$ for all $x' \in U$. Further, $F$ is upper semi-continuous (u.s.c.) if and only if for each $x \in X$ and for each $V = V^o \subset Y$ such that $F(x) \subseteq V$ there exists $U = U^o \subset X$ with $x \in U$ such that $F(U) \subseteq V$. The multifunction $F$ is continuous if and only if $F$ is both l.s.c. and u.s.c.

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If $r$ is a positive real number and if $A$ is a subset of a metric space $X$, then

$$S_r(A) = \{x | x \in X \text{ and } d(x, A) < r\}.$$  

**Definition.** Let $F: X \to Y$, where $Y$ is a metric space, and let $\varepsilon > 0$. Then $F$ is lower $\varepsilon$-continuous (l.\varepsilon-c.) if and only if for each $x \in X$ and for each $y \in F(x)$ there exists $U = U^o \subset X$ with $x \in U$ such that $F(x') \cap S_\varepsilon(y) \neq \emptyset$ for all $x' \in U$. Further, $F$ is upper $\varepsilon$-continuous (u.\varepsilon-c.) if and only if for each $x \in X$ there exists $U = U^o \subset X$ with $x \in U$ such that $F(U) \subset S_\varepsilon(F(x))$. The multifunction $F$ is $\varepsilon$-continuous if and only if $F$ is both l.\varepsilon-c. and u.\varepsilon-c.

Let $P$ be a property of sets. Then $F: X \to Y$ is point $P$ (image $P$) if and only if $F(x)$ has property $P$ for each $x \in X(F(E)$ has property $P$ for each $E \subset X$ with property $P$).

An alternate definition of $\varepsilon$-continuity for point closed multifunctions $F: X \to Y$, where $Y$ is a metric space of finite diameter, can be formulated in terms of the Hausdorff metric [1] for the space $S(Y)$ of non-empty, closed subsets of $Y$ and the induced function $f: X \to S(Y)$. If, moreover, $F$ is point compact, then the two definitions of $\varepsilon$-continuity are essentially equivalent [3].

We now indicate the close correspondence between continuity and $\varepsilon$-continuity. It follows from the above definitions that if $F: X \to Y$ is point compact and u.\varepsilon-c. for all $\varepsilon > 0$, then $F$ is u.\varepsilon-c. Furthermore, if $F: X \to Y$ is point compact, then $F$ is $\varepsilon$-continuous for all $\varepsilon > 0$ if and only if $F$ is continuous.

**Definition.** Let $A \subset X$. Then $A$ is an $m$-retract of $X$ if and only if there exists a continuous multifunction $F: X \to A$ such that $F(x) = \{x\}$ for all $x \in A$.

The proof of the following lemma is not difficult:

**Lemma 1.** Let $A$ be a compact subset of a metric space $X$, let $F: X \to A$ be an u.\varepsilon-c. multifunction such that $F(x) = \{x\}$ for all $x \in A$, and let $\eta > 0$. Then there is a $\lambda > 0$ such that if $x \in X$ and if $d(x, A) < \lambda$, then $d(x, F(x)) < \eta$ and $d(F(x)) < \eta$.

If $F: X \to Y$ and if $G: Y \to Z$, then a multifunction $G \circ F: X \to Z$ is defined by $G \circ F(x) = G(F(x))$ for each $x \in X$.

**Lemma 2.** If $F: X \to Y$ is $\varepsilon$-continuous where $Y$ is a subspace of a metric space $Z$ and if $G: Y \to Z$ is such that $d(y, z) \leq \varepsilon_0$ for all $y \in Y$ and for all $z \in G(y)$, then $G \circ F: X \to Z$ is $\varepsilon'$-continuous where $\varepsilon' = \varepsilon + 2\varepsilon_0$.

**Proof.** In order to show lower $\varepsilon'$-continuity, let $x \in X$ and let $z \in G \circ F(x)$. Then there exists $y \in F(x)$ such that $z \in G(y)$. Since $F$ is l.\varepsilon-c. there exists $U = U^o \subset X$ with $x \in U$ such that $F(x') \cap S_\varepsilon(y) \neq \emptyset$ for all $x' \in U$. Let $x' \in U$ and choose $y' \in F(x')$ such that $d(y, y') < \varepsilon$. Then
we have \( d(z, G(y')) \leq d(z, y) + d(y, y') + d(y', G(y')) < \varepsilon + 2\varepsilon_0 = \varepsilon' \). Therefore \( G \circ F(x') \cap S_\varepsilon(z) = \emptyset \), whence \( G \circ F \) is \( 1, \varepsilon' \)-c. The proof of upper \( \varepsilon' \)-continuity is entirely similar.

**Lemma 3.** If \( F : X \to Y \) is continuous and if \( G : Y \to Z \) is \( \varepsilon \)-continuous, then \( G \circ F : X \to Z \) is \( \varepsilon \)-continuous.

**Proof.** The proof of lower \( \varepsilon \)-continuity is straightforward. For upper \( \varepsilon \)-continuity let \( x \in X \). Then for each \( y \in F(x) \) there exists \( V_y = V_y^c \subset Y \) with \( y \in V_y \) such that \( G(V_y) \subset S_\varepsilon(G(y)) \) since \( G \) is \( u.e.c. \) As \( F \) is u.s.c. there exists \( U = U^c \subset X \) with \( x \in U \) such that \( F(U) \subset \bigcup \{ V_y \mid y \in F(x) \} \). It is clear that \( G \circ F(U) \subset S_\varepsilon(G \circ F(x)) \), whence \( G \circ F \) is \( u.e.c. \).

**Definition.** A metric space \( X \) has the **proximate fixed-point property for multifunctions** (p.F.p.p.) if and only if for each \( \eta > 0 \) there is a point \( x \in X \) such that \( d(x, F(x)) < \eta \).

The proximate fixed-point properties for single-valued functions (p.f.p.p.), for \( u.e.c. \) multifunctions, and for multifunctions with restrictions on the image sets are defined analogously.

**Theorem 1.** If a metric space \( X \) has the p.F.p.p., then every compact \( m \)-retract of \( X \) has the p.F.p.p.

**Proof.** Let \( A \) be a compact \( m \)-retract of \( X \) and let \( \eta > 0 \). Then there exists a continuous multifunction \( F : X \to A \) such that \( F(x) = \{ x \} \) for all \( x \in A \). By Lemma 1 there is a \( \lambda \in (0, \eta/8) \) such that if \( x \in X \) and if \( d\{ x, A \} < \lambda \), then \( d\{ x, F(x) \} < \eta/4 \) and \( d\{ F(x) \} < \eta/4 \). Since \( X \) has the p.F.p.p. there is an \( \varepsilon > 0 \) such that for every \( \varepsilon \)-continuous multifunction \( H : X \to X \) there is a point \( x \in X \) such that \( d\{ x, H(x) \} < \lambda \). The claim is that \( \varepsilon \) works for \( A \) also. For let \( G : A \to A \) be \( \varepsilon \)-continuous and let \( i \) be the inclusion on \( A \) into \( X \). Lemmas 2 and 3 readily imply that \( i \circ (G \circ F) : X \to X \) is \( \varepsilon \)-continuous. Thus there is a point \( x \in X \) such that \( d\{ x, G \circ F(x) \} < \lambda \). Consequently \( d\{ x, A \} < \lambda \), and therefore \( d\{ x, F(x) \} < \eta/4 \) and \( d\{ F(x) \} < \eta/4 \). Now we have \( d\{ F(x), G \circ F(x) \} \leq d\{ F(x), x \} + d\{ x, G \circ F(x) \} \leq \eta/4 + \lambda < \eta/2 \). Accordingly there exist points \( p, r \in F(x) \times F(x) \) such that \( d(r, q) < \eta/2 \). Thus \( d\{ p, G(p) \} \leq d\{ p, q \} < d\{ p, r \} + d\{ r, q \} < d\{ F(x) \} + \eta/2 < \eta \), and \( A \) has the p.F.p.p.

The above proof of Theorem 1 follows to a fashion of Klee's proof of the assertion of Theorem 1 for single-valued functions [2].

**Definition.** A space \( X \) has the **fixed-point property for multifunctions** (F.p.p.) if and only if for each continuous multifunction \( F : X \to X \) there is a point \( x \in X \) such that \( x \in F(x) \).

The fixed-point properties for single-valued functions (f.p.p.), for \( u.s.c. \) multifunctions, and for multifunctions with restrictions on the image sets are defined similarly.
Lemma 4. If a compact metric space $X$ has the p.F.p.p., then $X$ has the F.p.p. for point-closed multifunctions.

Proof. Let $F: X \to X$ be continuous and point closed. It is immediate that $G(F) = \bigcup \{(x, y)|x \in X$ and $y \in F(x)\}$ is a compact subset of $X \times X$. Therefore the function $d_{|\partial X}: G(F) \to R$, where $R$ denotes the real numbers, assumes a minimum value $r$ on $G(F)$. If $d(x, F(x)) > 0$ for each $x \in X$, then $r > 0$. Let $\eta = r/2$. Since $F$ is $\varepsilon$-continuous for every $\varepsilon > 0$, there does not exist an $\varepsilon > 0$ such that for every $\varepsilon$-continuous multifunction $G: X \to X$ there is a point $x \in X$ such that $d(x, G(x)) < \eta$.

A proof of Lemma 4 employing nets appears in [4] and [5]. Yandl [7] proved Lemma 4 for single-valued functions. Partial converses to this lemma may be found in [3] and [4].

Corollary. If a metric space $X$ has the p.F.p.p., then every compact $m$-retract of $X$ has the F.p.p. for point-closed multifunctions.

Let $E^2$ be a Euclidean 2-cell. By a theorem of Klee [2], every compact, metric absolute retract has the p.f.p.p. As a consequence $E^2$ has the p.f.p.p. but lacks the p.F.p.p. since Strother [6] has exhibited a continuous, point closed multifunction on $E^2$ into itself which does not have a fixed point.

Lemma 5. Let $g: Y \to Z$ be a continuous (single-valued) function on a compact metric space $Y$ into a metric space $Z$ and let $\varepsilon' > 0$. Then there is an $\eta > 0$ such that if $F: X \to Y$ is $\varepsilon$-continuous where $0 < \varepsilon \leq \eta$, then $g \circ F: X \to Z$ is $\varepsilon'$-continuous.

Proof. Since $Y$ is compact and $g$ is continuous, there is an $\eta > 0$ such that if $(y, y') \in Y \times Y$ and if $d(y, y') < \eta$, then $d(g(y), g(y')) < \varepsilon'$. Let $\varepsilon(0, \eta]$ and let $F: X \to Y$ be $\varepsilon$-continuous. To prove the lower $\varepsilon'$-continuity of $g \circ F$, let $x \in X$ and let $\varepsilon g \circ F(x)$. Then there is a point $y \in F(x)$ such that $g(y) = z$. Since $F$ is l.e-c. there exists $U = U^0 \subset X$ with $x \in U$ such that $F(x') \cap S_{\varepsilon}(y) = \emptyset$ for all $x' \in U$. Hence $g \circ F(x') \cap S_{\varepsilon}(z) = \emptyset$ for all $x' \in U$. Therefore $g \circ F$ is l.e-c. For upper $\varepsilon'$-continuity let $x \in X$. Since $F$ is u.e-c. there exists $U = U^0 \subset X$ with $x \in U$ such that $F(U) \subset S_{\varepsilon}(F(x))$. It follows that $g \circ F(U) \subset S_{\varepsilon}(g \circ F(x))$. Therefore $g \circ F$ is u.e-c., and hence $\varepsilon'$-continuous.

Theorem 2. If a compact metric space $X$ has the p.F.p.p., then every metric homeomorph of $X$ has the p.F.p.p.

Proof. Let $h: X \to Y$ be a homeomorphism of $X$ onto a metric space $Y$ and let $\eta > 0$. Since $X$ is compact and $h$ is continuous, there is an $\eta' > 0$ such that if $(x, x') \in X \times X$ and if $d(x, x') < \eta'$, then $d(h(x), h(x')) < \eta$. Since $X$ has the p.F.p.p. there is an $\varepsilon' > 0$ such that for every $\varepsilon'$-continuous multifunction $F: X \to X$ there is a point $x \in X$ such that $d(x, F(x)) < \eta'$. Since $Y$ is compact and $h^{-1}: Y \to X$ is a continuous function, there exists an $\varepsilon > 0$ by Lemma 5 such that if $F': X \to Y$ is
\( \varepsilon \)-continuous, then \( h^{-1} \circ F' : X \rightarrow X \) is \( \varepsilon' \)-continuous. Let \( G : Y \rightarrow Y \) be an arbitrary \( \varepsilon \)-continuous multifunction. Lemmas 3 and 5 imply that \( h^{-1} \circ (G \circ h) : X \rightarrow X \) is \( \varepsilon' \)-continuous. Thus there is a point \( x_0 \in X \) such that \( d(x, h^{-1} \circ (G \circ h)(x)) < \eta' \), whence \( d(h(x), G(h(x))) < \eta \) and \( Y \) has the p.F.p.p.

It can be seen by means of an inductive proof that \((0, 1)\) and \([0, 1]\) both have the p.F.p.p. for point compact multifunctions. Therefore the p.F.p.p. is not a topological invariant since the set of real numbers clearly does not have the p.F.p.p.

We now prove a generalization of the last theorem in Klee's paper [2].

**Theorem 3.** Let \( X \) be a compact Hausdorff space which is an absolute retract for such spaces. Then for each open cover \( \mathcal{U} \) of \( X \) there exists a finite open cover \( \mathcal{V} \) of \( X \) which has the following property: if \( G : X \rightarrow X \) is any multifunction such that for each \( x \in X \) there exists \( N_x = N_x^0 \subset X \) with \( x \in N_x \) satisfying \( G(N_x) \subset V \) for some \( V \in \mathcal{V} \), then there is a point \( x_0 \in X \) such that \( x_0 \in U \) and \( G(x_0) \subset U \) for some \( U \in \mathcal{U} \).

**Proof.** We can assume that \( X \) is a compact retract of a Tychonoff cube \( T = [0, 1]^M \), and we consider \( T \) as a subset of the linear topological space \( R^M \), where \( R \) denotes the real numbers. Let \( \mathcal{B} \) be a symmetric base for the uniformity for \( T \) such that for each \( x \in T \) and for each \( B \in \mathcal{B} \), we have that \( B[x] \) is convex. Suppose that \( \mathcal{U} \) is an open cover of \( X \). Then there exists a member \( R \) of the uniformity for \( T \) such that \( R[x] \) is a subset of some member of \( \mathcal{U} \) for every \( x \in X \). Choose \( \hat{B} \in \mathcal{B} \) such that \( \hat{B} \subset B \) and \( f(x) = x \) for all \( x \in X \). There exists \( \hat{B} \in \mathcal{B} \) with \( \hat{B} \subset B \) such that if \( t \in T \) and if \( \hat{B}[t] \cap X \neq \emptyset \), then \( f(t) \in B[t] \). This follows from an extension of Lemma 1. Choose \( B \in \mathcal{B} \) such that \( B \circ B \subset \hat{B} \). Then we let \( \mathcal{V} \) be a finite subcover of \( \{ B[x] | x \in X \} \) which covers \( X \). Suppose that \( G : X \rightarrow X \) is any multifunction such that for each \( x \in X \) there exists \( N_x = N_x^0 \subset X \) with \( x \in N_x \) satisfying \( G(N_x) \subset V \) for some \( V \in \mathcal{V} \). Now choose a finite subcover \( \mathcal{N} \) of \( \{ N_x | x \in X \} \) which covers \( X \), and for each \( x \in X \) let \( W_x = \bigcap \{ N_x^0 | x \in X \} \). For each \( S \subset X \) let \( CH(S) \) denote the convex hull of \( S \) in \( T \). Then define a multifunction \( H : X \rightarrow T \) by \( H(x) = (CH(G(W_x)))^0 \) for each \( x \in X \). The multifunction \( H \) is obviously point closed, point convex, and u.s.c. By the Kakutani-Fan-Glicksberg fixed-point theorem [1] (which is generalized in [3] and [4]), there is a point \( t_0 \in T \) such that \( t_0 \in H \circ f(t_0) \). Let \( x_0 = f(t_0) \). By the definition of \( H \) there is a point \( t_1 \in CH(G(W_{x_0})) \) such that \( t_1 \in B[t_0] \). Since \( \mathcal{N} \) covers \( X \) and since \( B[x] \) is convex for each \( x \in T \), there is a point \( (x, x') \in X \times X \) such that \( CH(G(W_{x_0})) = CH(G(N_{x})) \subset B[x] \). Thus \( (t_0, t) \in \hat{B} \), whence \( (t_0, x_0) \in \hat{B} \). Therefore \( (x, x_0) \in R \). But also \( G(x_0) \subset B[x] \subset R[x] \), and this proves the theorem.
Remarks. Theorems 1 and 2 are easily generalized to uniform spaces. Furthermore Theorem 1 can be extended to a generalization of a theorem of Yandl [7] on strong proximate retracts. It is only necessary to use Lemma 1 as a guide to defining the concept of a strong proximate \( m \)-retract thereby obtaining directly an extension of Theorem 1 and Yandl’s theorem. The details are in [3]. Variants of Theorems 1 and 2 are of course immediate by restricting the classes of multifunctions considered. We indicate in closing an outgrowth of Theorem 3. Suppose that \( X \) is a compact Hausdorff space which has a symmetric base \( \mathcal{B}_0 \) for its uniformity. For a given property \( P \) of sets, let \( P(X) \) be the subsets of \( X \) which have property \( P \). Suppose further that there is a function \( K : 2^X \to 2^X \) (where \( 2^X \) denotes the collection of all subsets of the set \( X \)) such that \( P \) and \( K \) satisfy the following conditions:

(i) If \( A \in 2^X \), then \( A \subseteq K(A) \cap P(X) \).

(ii) If \( A \subseteq B \subseteq 2^X \), then \( K(A) \subseteq K(B) \).

(iii) If \( A \in P(X) \), then \( K(A) = A \) and \( A^* \in P(X) \).

(iv) \( \{[x] | x \in X \} \cup \{\emptyset\} \subseteq P(X) \).

(v) If \( A \in P(X) \) and if \( U \in \mathcal{B}_0 \), then \( U[A] \in P(X) \).

We call \( K \) a \( P \)-operator for \( X \). If, in addition to the above assumptions, \( X \) has the F.p.p. for point closed, point \( P \), u.s.e. multifunctions — then the conclusion of Theorem 3 is verified. The proof of this assertion duplicates the proof of Theorem 3 given above. Two examples of spaces \( X \) with the aforementioned properties are non-empty, compact, convex subsets of locally convex, Hausdorff linear topological spaces and hereditarily unicoherent, arcwise connected, locally connected continua (trees) [4].

REFERENCES


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