RETRACTS OF THE PSEUDO-ARC

BY

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Professor B. Knaster has raised [3] the question as to whether the pseudo-arc has a non-trivial retract. Corollary 1 of this paper states that every subcontinuum of the pseudo-arc is a retract of the pseudo-arc. Descriptions and properties of the pseudo-arc may be found in [1], [2], [4], [5], and [6]. Except as noted, terminology used here related to chains is from [1] and [4] and that related to general topology is from [7].

The following theorem by R. H. Bing is of particular importance in the study of the pseudo-arc:

**Lemma 1.** Suppose \( x_1, x_2, \ldots, x_n \) is a collection of positive integers such that \( h = x_1 \leq x_i \leq x_n = k \) and

\[
|x_i - x_{i+1}| \leq 1 \quad (i = 1, 2, \ldots, n-1).
\]

Suppose also that \( D_1, D_2, \ldots \) is a sequence of chains from \( P \) to \( Q \) such that for each positive integer \( i \), \( D_{i+1} \) is crooked in \( D_i \), the closure of each link of \( D_{i+1} \) is a compact subset of a link of \( D_i \), and the mesh of \( D_i \) is less than \( 1/i \). Let \( d(i)_r \) denote the \( r \)-th link of \( D_i \). Suppose further that the subchain \( D_2(u, v) \) of \( D_2 \) is contained in the subchain \( D_1(h, k) \) of \( D_1 \) and the closures of \( d(2)_u \) and \( d(2)_v \) are mutually exclusive subsets of \( d(1)_h \) and \( d(1)_k \) respectively. Then for each integer \( w \) there is an integer \( j \) greater than \( w \) and a chain \( E = [e_1, e_2, \ldots, e_n] \) following the pattern \( (1, x_1), (2, x_2), \ldots, (n, x_n) \) in \( D_1 \) such that \( E \) is a consolidation of the links of \( D_j \) contained in \( D_2(u, v) \) and no interior link of \( E \) intersects \( d(2)_u + d(2)_v \).

If \( a \) is a chain and \( \beta \) is a subchain of \( a \), a retraction of \( a \) to \( \beta \) is a transformation \( \gamma \) from \( a \) to \( \beta \) which preserves adjacency such that if \( x \) is in \( \beta \), \( \gamma(x) \) is \( x \). The consolidation of \( a \) induced by \( \gamma \) is the chain to which a link \( y \) belongs if and only if for some \( x \) in \( \beta \), \( y \) is \([\gamma^{-1}(x)]^*\).

The usual description of the pseudo-arc is slightly altered. Specifically, it is assumed throughout that \( P \) and \( Q \) are two points of a compact metric space and that \( W_1, W_2, \ldots \) is a sequence of chains in that space from \( P \) to \( Q \) and \( A_1, A_2, \ldots \) and \( B_2, B_3, \ldots \) and \( C_2, C_3, \ldots \) are sequences of chains such that
(1) \( W_1 \) has nine links and \( A_1 \) is \( W_1 \) (1, 5).

(2) For each positive integer \( i \), the mesh of \( W_i \) is less than \( 1/i \) and any link of \( W_i \) that contains a link of \( W_{i+1} \) contains the closure of that link.

(3) For each integer \( i \) greater than 1, \( A_i \) and \( B_i \) have only last links in common, \( B_i \) and \( C_i \) have only first links in common, \( W_i \) is \( A_i + B_i + C_i \) (the order of \( B_i \) is reversed in \( W_i \)), \( A_i \) and \( B_i \) are crooked in \( A_{i-1} \) and have a common pattern in \( A_{i-1} \), \( C_i \) is crooked in \( W_{i-1} \), the first links of \( A_i \) and \( B_i \) are in only the first link of \( A_{i-1} \) (which is also the first link of \( W_{i-1} \)), and the last links of \( A_i \) and \( C_i \) are in only the last links of \( A_{i-1} \) and \( W_{i-1} \) respectively (see figure).

Let \( M \) denote the intersection of \( W_1^*, W_2^*, \ldots \) and let \( R \) denote the intersection of \( A_1^*, A_2^*, \ldots \). It is not difficult to verify that for each positive integer \( i \), \( W_{i+1} \) is crooked in \( W_i \) so that \( M \) is the pseudo-arc. Furthermore, \( R \) is a non-degenerate proper subcontinuum of \( M \) and it will be shown that

**Theorem 1.** \( R \) is a retract of \( M \).

We must describe a continuous transformation \( \theta \) from \( M \) to \( R \) such that for each \( x \) in \( R \), \( \theta(x) \) is \( x \). In so doing, we will use methods analogous to those of Lelek in [5].

**Proof of Theorem 1.** We define, by induction, an infinite sequence 
\[ \{[n(i), V_i, R_i, \theta_i, E_i]\}_{i=1}^\infty \]

such that \( \{n(i)\}_{i=1}^\infty \) is an increasing sequence of integers with \( n(1) = 1 \), and for \( i = 1, 2, \ldots \)

(a) \( V_i \) is a consolidation of \( W_{n(i)} \) which is a chain from \( P \) to \( Q \);

(b) \( R_i \) is an initial subchain of \( V_i \) which contains \( A_{n(i)} \) and (for \( i > 1 \)) refines \( A_{n(i-1)} \) — specifically, \( R_1 \) is \( A_1 \) and for \( i > 1 \), \( R_i \) is the chain such that \( A \) is a link of it if and only if for some link \( A' \) of \( A_{n(i-1)} \), \( A \) is the sum of the links of \( W_{n(i)} \) that lie in \( A' \);

(c) \( \theta_i \) is a retraction of \( V_i \) to \( R_i \) which takes the last link of \( V_i \) to the last link of \( R_i \).
(d) if \( a \) is a link of \( V_{i+1} \), there is a link \( \beta \) of \( R_i \) such that \([\theta_i^{-1}(\beta)]^*\) contains \( a \) and \( \beta \) contains \( \theta_{i+1}(a) \); and

(e) \( E_i \) is the consolidation of \( V_i \) induced by \( \theta_i \).

For any such sequence, \( M \) is the intersection of \( V_1^*, V_2^*, \ldots \) and \( R \) is the intersection of \( R_1^*, R_2^*, \ldots \). In the process, additional sequences, \( \{S_i\}_{i=2}^\infty \) and \( \{T_i\}_{i=2}^\infty \) of chains will appear.

*Initial step.* Let \( n(1) = 1 \), \( V_1 \) be \( W_1, R_1 \) be \( A_1 \) and let \( \theta_1 \) be the retraction of \( V_1 \) to \( R_1 \) that takes each link of \( V_1 - R_1 \) to the last link of \( R_1 \). Let \( E_1 \) be the consolidation of \( V_1 \) induced by \( \theta_1 \). The parts of conditions (a)-(e) that are applicable are readily verified.

*Induction step.* Suppose \( p \) is a positive integer and \( \{(n(i), V_i, R_i, \theta_i, E_i)\}_{i=1}^p \) have been defined and satisfy conditions (a)-(e), where applicable. Consider Lemma 1 with the following designations:

(i) \( x_1, x_2, \ldots, x_n \) describes a common pattern of \( A_{n(p)+1} \) and \( B_{n(p)+1} \) in \( R_p \) (such a pattern exists because \( R_p \) contains \( A_{n(p)} \) and \( A_{n(p)+1} \) and \( B_{n(p)+1} \) have a common pattern in \( A_{n(p)} \)).

(ii) \( D_1 \) is \( E_p \), \( D_i \) is \( W_{n(p)+i-1}, i = 2, 3, \ldots \)

(iii) \( D^p_2(u, v) \) is \( C_{n(p)+1} \) and \( D_4(h, k) \) is \( E_p \).

From the specifications in (b) and the definition of \( M \), it follows that \( x_1 \) is 1 and \( x_n \) is the number of links in \( R_p \). The fact that the mesh of \( D_1 \) is possibly not less than 1/1 is not significant. From (b), (c) and (e) it follows that the first link of \( E_p \) contains the first link of \( V_p \), and from (e) and (e) it follows that the last link of \( E_p \) contains the last link of \( V_p \). From (a) it follows that the first and last links of \( V_p \) contain, respectively, the first and last links of \( W_{n(p)} \). From the definition of \( M \), the first and last links of \( W_{n(p)} \) are disjoint and contain, respectively, the closures of the first and last links of \( C_{n(p)+1} \). We conclude that the closures of the first and last links of \( C_{n(p)+1} \) (\( d(2)_u \) and \( d(2)_v \)) are mutually exclusive subsets of the first and last links of \( E_p \) (\( d(1)_h \) and \( d(1)_k \)) respectively. The remainder of the hypothesis of Lemma 1 is easily verified.

Consequently, there is an integer \( n(p+1) \) greater than \( n(p) \) and a chain \( T_{p+1}^\prime \) following the pattern \((1, x_1), (2, x_2), \ldots, (n, x_n)\) in \( E_p \) such that \( T_{p+1} \) is a consolidation of the links of \( W_{n(p+1)} \) contained in \( C_{n(p)+1} \) and no interior link of \( T_{p+1} \) intersects an end link of \( C_{n(p)+1} \). Let \( R_{p+1}^\prime \) be the chain such that \( \Lambda \) is a link of it if and only if for some link \( \Lambda' \) of \( A_{n(p)+1} \), \( \Lambda \) is the sum of the links of \( W_{n(p+1)} \) that lie in \( \Lambda' \). Let \( S_{p+1} \) be the chain such that \( \Lambda \) is a link of it if and only if \( \Lambda \) is the first link of \( T_{p+1} \) or for some link \( \Lambda' \) of \( B_{n(p)+1} \) other than the first link, \( \Lambda \) is the sum of the links of \( W_{n(p+1)} \) that lie in \( \Lambda' \). Let \( S_{p+1} \) be the chain such that \( \Lambda \) is a link of it if and only if \( \Lambda \) is the first link of \( T_{p+1} \) or for some link \( \Lambda' \) of \( B_{n(p)+1} \) other than the first link, \( \Lambda \) is the sum of the links of
$W_{n(p+1)}$ that lie in $\cal A'$. Because both $A_n(p+1)$ and $B_n(p+1)$ have the pattern $(1, x_1), (2, x_2), \ldots, (n, x_n)$ in $R_p$, $R_{p+1}$ has the same pattern in $E_p$. Let $V_{p+1}$ be $R_{p+1} + S_{p+1} + T_{p+1}$ and let $\theta_{p+1}$ be the retraction of $V_{p+1}$ to $R_{p+1}$ that takes the $r$-th link of $S_{p+1}$ and the $r$-th link of $T_{p+1}$ to the $r$-th link of $R_{p+1}$. Let $E_{p+1}$ be the consolidation of $V_{p+1}$ induced by $\theta_{p+1}$.

(a) $V_{p+1}$ is obviously a consolidation of $W_{n(p+1)}$. The first link of $V_{p+1}$ is the first link of $R_{p+1}$, contains the first link of $W_{n(p+1)}$ and therefore contains $P$. $Q$ belongs to the last link of $W_{n(p+1)}$ which is contained in the last link of $C_{n(p+1)}$ and $Q$ must then belong to a link of $T_{p+1}$. Because $x_1 = 1$, the first link of $T_{p+1}$ is a subset of the first link of $C_{n(p+1)}$; from the conclusion of Lemma 1, no interior link of $T_{p+1}$ intersects an end link of $C_{n(p+1)}$. Consequently, the last link of $T_{p+1}$, which is the last link of $V_{p+1}$, contains $Q$. It follows that $V_{p+1}$ is a chain from $P$ to $Q$.

(b) The specifications are contained in the definition of $R_{p+1}$ and imply that $R_{p+1}$ refines $A_n(p)$ and contains $A_{n(p+1)}$. That $R_{p+1}$ is an initial subchain of $V_{p+1}$ follows from the definition of $V_{p+1}$.

(c) By definition, $\theta_{p+1}$ is a retraction of $V_{p+1}$ to $R_{p+1}$ and because $x_n$ is the number of links of $R_p$, $\theta_{p+1}$ takes the last link of $T_{p+1}$ (which is the last link of $V_{p+1}$) to the last link of $R_{p+1}$.

(d) Suppose $a$ is a link of $V_{p+1}$. If $a$ is the $r$-th link of $R_{p+1}$ or the $r$-th link of $T_{p+1}$, each of $a$ and $\theta_{p+1}(a)$ (which is the $r$-th link of $R_{p+1}$) is a subset of the $x_r$-th link $\beta$ of $R_p$ and since $\theta_p(\beta)$ is $\beta$, $[\theta_p^{-1}(\beta)]^*$ contains $a$ and $\beta$ contains $\theta_{p+1}(a)$. If $a$ is the $r$-th link of $T_{p+1}$, let $\beta$ denote the $x_r$-th link of $R_p$. Then $[\theta_p^{-1}(\beta)]^*$ is the $x_r$-th link of $E_p$ and must contain $a$, and $\beta$ contains the $r$-th link of $R_{p+1}$ which is $\theta_{p+1}(a)$.

(e) Satisfied by definition of $E_{p+1}$.

This completes the induction step.

**Description of $\theta$.** For each point $x$ of $M$ and positive integer $i$, let $K_i(x)$ be the link or links (at most two) of $v_i$ containing $x$, and let $J_i(x)$ be the sum of the elements of $\theta_i[K_i(x)]$. For $x$ in $M$ and $i = 2, 3, \ldots, J_i(x)$ is either a link of $R_i$ or the sum of two intersecting links of $R_i$, and because $R_i$ refines $A_{n(i-1)}$, the diameter of $J_i(x)$ is less than $2/(i-1)$. If $a$ is in $K_{i+1}(x)$, there is a link $\beta$ of $R_i$ such that $[\theta_{i}^{-1}(\beta)]^*$ contains $a$ and $\beta$ contains $\theta_{i+1}(a)$, and there is a link $\lambda$ of $\theta_i^{-1}(\beta)$ that contains $x$ and hence belongs to $K_i(x)$. Then $\theta_i(\lambda)$ contains $\theta_{i+1}(a)$ and we conclude that $J_i(x)$ contains $J_{i+1}(x)$.

Consequently, for $x$ in $M$, $J_1(x), J_2(x), \ldots$ is a monotonic sequence of compact sets whose diameters converge to zero and we define $\theta(x)$ to be the one point common to all of $J_1(x), J_2(x), \ldots$
$\theta$ is a retraction. If $x$ is in $R$, for $i = 1, 2, \ldots, K_i(x)$ includes one or two links of $R_i$ so that $x$ is in $J_i(x)$. Consequently, if $x$ is in $R$, $\theta(x)$ is $x$.

$\theta$ is continuous. Suppose $\varepsilon$ is a positive number. Let $i > 1$ be an integer such that $3/(i-1) < \varepsilon$, and let $\delta$ be the Lebesgue number of the open cover $V_i$ of $M$. Suppose $x$ and $y$ belong to $M$ and the distance from $x$ to $y$ is less than $\delta$. Then some link of $V_i$ contains both $x$ and $y$ so that $K_i(x) + K_i(y)$ has at most three links and the diameter of $J_i(x) + J_i(y)$ is less than $3/(i-1) < \varepsilon$. Since $\theta(x)$ is in $J_i(x)$ and $\theta(y)$ is in $J_i(y)$, the distance from $\theta(x)$ to $\theta(y)$ is less than $\varepsilon$. It follows then that $\theta$ is continuous and the proof of Theorem 1 is complete.

Corollary 1. Every subcontinuum of $M$ is a retract of $M$.

Suppose $S$ is a subcontinuum of $M$. If $S$ is degenerate or $M$, $S$ is a trivial retract of $M$. Suppose then that $S$ is a non-degenerate proper subcontinuum of $M$. From Theorem 15 of [1], there is a homeomorphism $H$ from $M$ to $M$ such that $H(M)$ is $M$ and $H(R)$ is $S$. Then the composition of $H$ restricted to $R$ and $\theta$ and $H^{-1}$ is a retraction of $M$ to $S$.

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REFERENCES


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