COLLOQUIUM MATHEMATICUM

VOL. XIX 1968 FASC. 1

ON THE EXISTENCE OF σ-COMPLETE PRIME IDEALS IN BOOLEAN ALGEBRAS

BY

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1. Introduction. Recently, the author posed the following problem. Does there exists a complete non-atomic Boolean algebra which has a σ-complete prime ideal? It appeared as problem P 461 in the problem section of the journal Colloquium Mathematicum 14 (1964), p. 148.

In the present paper* we shall give a partial answer to this question. First of all we shall show that if $\mathscr{B}$ is a non-atomic complete Boolean algebra such that every mutually disjoint subset of $\mathscr{B}$ has a cardinal which is not measurable, then every prime ideal is not σ-complete. We recall that a cardinal a is called *measurable* whenever the Boolean algebra of all subsets of a has a non-principal σ-complete prime ideal. Furthermore, we shall single out a special class of non-atomic complete Boolean algebras for which it can be shown directly that their prime ideals are not σ-complete.

2. The existence of σ-complete prime ideals. For terminology and notation not explained in this paper we refer to [7].

In this paper we shall only consider non degenerate Boolean algebras. The elements of a Boolean algebra $\mathscr{B}$ will be denoted by $a, b, \ldots$ with or without subscripts; the zero element by 0 and the unit element by 1. The Boolean operations of join and meet will be denoted by $\vee$ and $\wedge$, respectively. The unique complement of an element $a \in \mathscr{B}$ will be denoted by $-a$.

An ideal $I \subset \mathscr{B}$ is called *σ-complete* whenever $A \subset I$, A is countable and $\sup A$ exists implies $\sup A \in I$. An ideal which is not σ-complete will be called σ-incomplete. An ideal I is called *prime* whenever $a \wedge b \in I$ implies $a \in I$ or $b \in I$. An ideal is prime if and only if it is maximal, i.e., if it is not properly contained in an ideal. An element $a \in \mathscr{B}$ is called an *atom* whenever $0 \leqslant b \leqslant a$ implies $b = 0$ or $b = a$.

* Work on this paper was supported in part by grant No. GP-6111 from the National Science Foundation.

If the Boolean algebra $\mathscr{B}$ has an atom a, then the prime ideal I consisting of all b with $b \wedge a = 0$ is obviously σ-complete, and, in fact is *complete*, i.e., for every non-empty subset $A \subset I$ if $\sup A$ exists, then $\sup A \in I$. More precisely, a complete Boolean algebra $\mathscr{B}$ has a complete prime ideal if and only if $\mathscr{B}$ has an atom. Hence, if $\mathscr{B}$ is non-atomic and complete it is natural to ask whether $\mathscr{B}$ has a σ-complete prime ideal. Before we shall give a few answers to this question we shall first present a few preliminaries and a lemma.

Let $\mathscr{B}$ be a Boolean algebra and let $I \subset \mathscr{B}$ be an ideal in $\mathscr{B}$. Then I is called *dense* in $\mathscr{B}$ whenever for every $0 < a \in \mathscr{B}$ there is an element $b \in I$ such that $0 < b \leqslant a$. In that case, however, I has a stronger property namely

$$(*) \qquad a = \sup(b : b \in I \text{ and } b \leqslant a)$$

for all $a \in \mathscr{B}$. Indeed, if I is dense and (*) does not hold for some $a \in \mathscr{B}$, then there is an element $a_1 < a$ such that $b \in I$, $b \leqslant a$ implies $b \leqslant a_1$, and so $(a - a_1) \wedge b = 0$ for all $b \in I$, which is impossible since I is dense.

We shall now prove the following lemma:

LEMMA 1. *If $\mathscr{B}$ is a non-atomic Boolean algebra, then every prime ideal $I \subset \mathscr{B}$ is dense.*

Proof. If the prime ideal $I \subset \mathscr{B}$ is not dense in $\mathscr{B}$, then there is an element $0 < a \in \mathscr{B}$ such that $a \wedge I = 0$. Since $\mathscr{B}$ has no atoms it follows that there exists an element $0 < b < a$, and so $I \vee I_b$ where I_b is the principal ideal generated by b, is an ideal which properly contains I; contradicting the maximality of I and the proof is finished.

The main result of the paper is given in the following theorem:

THEOREM 1. *Assume that there are no measurable cardinals and assume that $\mathscr{B}$ is a non-atomic complete Boolean algebra. Then every prime ideal in $\mathscr{B}$ is σ-incomplete.*

Proof. Assume on the contrary that there does exist a Boolean algebra $\mathscr{B}$ which is non-atomic, complete and which has a σ-complete prime ideal I. From Lemma 1 it follows that I is dense in $\mathscr{B}$. Let $\{a_\sigma : \sigma \in \Sigma\}$ be a maximal disjointed (i.e., $\sigma_1 \neq \sigma_2$ implies $a_{\sigma_1} \wedge a_{\sigma_2} = 0$) subset of I. Then $\sup(a_\sigma : \sigma \in \Sigma) = 1$. Indeed, if $\sup(a_\sigma : \sigma \in \Sigma) = a_0 \neq 1$, then I being dense there is an element $a \in I$ such that $0 < a < -a_0$ which contradicts the maximality of $\{a_\sigma : \sigma \in \Sigma\}$. We shall now prove that Σ is measurable. To this end, we set $X \in J$ whenever $X \subset \Sigma$ and $\sup(a_\sigma : \sigma \in X) \in I$. It is easy to see that J is an ideal of the algebra of all subset of Σ which is proper since $\sup(a_\sigma : \sigma \in \Sigma) = 1 \notin I$. Furthermore, $a_\sigma \in I$ implies $\{\sigma\} \in J$ for all $\sigma \in \Sigma$. Finally it is easy to check that I is prime and σ-complete implies J is prime and σ-complete. Thus Σ is measurable, which contradicts the hypothesis and the proof is finished.

Let $\mathscr{B}$ be a Boolean algebra. Then by $c(\mathscr{B})$ we shall denote the smallest cardinal with the property that the cardinal of every disjointed system in $\mathscr{B}$ is bounded by $c(\mathscr{B})$. With this definition the following result is an immediate consequence of the proof of Theorem 1:

COROLLARY 1. *Let $\mathscr{B}$ be a non-atomic complete Boolean algebra such that $c(\mathscr{B})$ is non-measurable. Then every prime ideal in $\mathscr{B}$ is σ-incomplete.*

Remark. It is known that it is consistent with the axioms of set theory to assume that measurable cardinals do not exist and so it is also consistent to assume that every prime ideal in a non-atomic complete Boolean algebra is σ-incomplete. It was shown by Ulam [9] that all the cardinals less than the first inaccesible cardinal are non-measurable and more recently Tarski [8] proved the interesting result that even a large number of inaccessible cardinals are non-measurable. Thus except possibly for very exceptional complete non-atomic Boolean algebras the prime ideals in such algebras are σ-incomplete.

We do not know whether the existence of a measurable cardinal implies the existence of a complete non-atomic Boolean algebra which has a σ-complete prime ideal.

3. Boolean algebras with continuous resolutions. Let $\mathscr{B}$ be a Boolean algebra and let $0 < a \in \mathscr{B}$. Then a is said to have a *continuous resolution* whenever there is a mapping $t \to a(t)$ of $0 \leqslant t \leqslant 1$ into $\mathscr{B}$ such that

$$0 = a(0) = \inf\big(a(t) : 0 < t \leqslant 1\big); \qquad a = a(1) = \sup\big(a(t) : 0 \leqslant t < 1\big);$$

$0 \leqslant t_1 \leqslant t_2 \leqslant 1$ implies $a(t_1) \leqslant a(t_2)$; and for all $0 < t_0 < 1$,

$$a(t_0) = \sup\big(a(t) : 0 \leqslant t < t_0\big) = \inf\big(a(t) : t_0 < t \leqslant 1\big).$$

A Boolean algebra $\mathscr{B}$ is said to have a *continuous resolution* whenever its unit element has a continuous resolution. In that case, every element of $\mathscr{B}$ has a continuous resolution. Of course, if some element $0 < a \in \mathscr{B}$ with $a \neq 1$ has a continuous resolution, then $\mathscr{B}$ need not have a continuous resolution.

Boolean algebras with a continuous resolution have the following property:

THEOREM 2. *If a Boolean algebra $\mathscr{B}$ has a continuous resolution, then every prime ideal of $\mathscr{B}$ is σ-incomplete. In particular, $\mathscr{B}$ is non-atomic.*

Proof. Let $\{e(t) : 0 \leqslant t \leqslant 1\}$ be a continuous resolution of $\mathscr{B}$ and let $I \subset \mathscr{B}$ be a σ-complete ideal. Then the set

$$E = \{t : 0 \leqslant t \leqslant 1 \text{ and } e(t) \in I\}$$

is non-empty $(0 \in E)$ and is bounded from above by 1, and so $t_0 = \sup E$ exists. Since $\{e(t) : 0 \leqslant t \leqslant 1\}$ is a continuous resolution and I is σ-complete we obtain from

$$e(t_0) = \sup\big(e(t_n) : t_n \uparrow t_0\big)$$

that $e(t_0) \in I$. On the other hand, since I is prime we have that $-e(t) \in I$ for all $t > t_0$, and so if $t_n \downarrow t_0$ we see that

$$-e(t_0) = \sup\big(-e(t_n) : t_n \downarrow t_0\big) \in I.$$

Hence $1 = e(t_0) \vee -e(t_0) \in I$ and a contradiction is obtained.

If $\mathscr{B}$ would have an atom, then it would also have a complete prime ideal. This completes the proof of the theorem.

We recall that a point x of a topological space X is called a *P-point* whenever the intersection of every countable family of neighborhoods of x is a neighborhood of x. Every discrete point of a topological space is of course a P-point. Furthermore, if $S(\mathscr{B})$ denotes the Stone representation space of a Boolean algebra $\mathscr{B}$, then $x \in S(\mathscr{B})$ is a P-point if and only if the corresponding prime ideal in $\mathscr{B}$ is σ-complete. The following result is then evident:

COROLLARY 2. *The Stone representation space of a Boolean algebra which has a continuous resolution has no P-points.*

In the next section we shall single out an important subclass of Boolean algebras with continuous resolutions.

4. Hyperstonian Boolean algebras. Let $\mathscr{B}$ be a Boolean algebra. A real function μ defined on $\mathscr{B}$ is called a *state* whenever μ has the following properties: $\mu(a) \geqslant 0$ for all $a \in \mathscr{B}$; $\mu(a \vee b) = \mu(a) + \mu(b)$ for all $a, b \in \mathscr{B}$ satisfying $a \wedge b = 0$; and $\mu(1) \neq 0$. A state is called *strictly positive* whenever $\mu(a) = 0$ implies $a = 0$. Every state is monotone increasing and a strictly positive state is strictly increasing.

A state μ is called a *measure*, whenever μ is countably additive, i.e., $a_n \downarrow 0$ ($\{a_n\}$ is decreasing and $\inf a_n = 0$) implies $\mu(a_n) \downarrow 0$. A state μ is called *pure* whenever $0 \leqslant \nu \leqslant \mu$ and ν is a measure implies $\nu = 0$. Every state can be written uniquely as a sum of a pure state and a measure.

A state μ is called *normal*, whenever $a_\tau \downarrow 0$ ($\{a_\tau\}$ is directed downwards and $\inf a_\tau = 0$) implies $\mu(a_\tau) \downarrow 0$. Every normal state is a measure. A measure μ is called *singular* whenever $0 \leqslant \nu \leqslant \mu$ and ν is normal implies $\nu = 0$. Every state μ can be written uniquely as the sum of a pure state and a singular measure and a normal state.

Following Dixmier [2], we shall call a complete Boolean algebra *hyperstonian* whenever for every $0 < a \in \mathscr{B}$ there is a normal state μ on $\mathscr{B}$ such that $\mu(a) \neq 0$. The connection being that a compact Hausdorff space which is hyperstonian in the sense of Dixmier is the Stone representation space of the hyperstonian Boolean algebra of all its open and closed subsets.

It is our purpose to show that non-atomic hyperstonian Boolean algebras have continuous resolutions. Before we shall turn to the proof of this result we shall first present some preliminary results.

Let $\mathscr{B}$ be a Boolean algebra and let μ be a state defined on $\mathscr{B}$. Then a is called a *μ-atom* whenever $\mu(a) > 0$ and $0 \leqslant b \leqslant a$ implies either $\mu(a) = \mu(b)$ or $\mu(b) = 0$. If a is an atom of $\mathscr{B}$ and $\mu(a) > 0$, then a is a μ-atom of $\mathscr{B}$. Of course a μ-atom need not be an atom. If μ is strictly positive, however, then an element is an atom if and only if it is a μ-atom. In this direction we have also the following less trivial result:

LEMMA 2. *If $\mathscr{B}$ is a complete Boolean algebra and if μ is a normal state defined on $\mathscr{B}$, then a is a μ-atom implies a is an atom. In particular, the normal states on a non-atomic complete Boolean algebra are free of atoms.*

Proof. Let μ be a normal state and let $0 < a \in \mathscr{B}$ be a μ-atom. Let

$$a_0 = \sup\big(b : 0 \leqslant b \leqslant a \text{ and } \mu(b) = 0\big).$$

Then, since μ is normal, $\mu(a_0) = 0$. We shall prove that $a - a_0$ is an atom of $\mathscr{B}$. From $\mu(a) > 0$ and $\mu(a_0) = 0$ it follows that $a - a_0 \neq 0$. Furthermore, if $0 < b \leqslant a - a_0$, then $\mu(b) = \mu(a)$ since a is a μ-atom. Thus $\mu(a - a_0 - b) = 0$ implies $a - a_0 - b \leqslant a_0$, i.e., $b = a - a_0$. This completes the proof.

Let $\mathscr{B}$ be a Boolean algebra and let μ be a state defined on $\mathscr{B}$. A mapping $t \to a(t)$ of $0 \leqslant t \leqslant 1$ into $\mathscr{B}$ is called a *μ-resolution* of a whenever $a(0) = 0$; $e(1) = 1$; $0 \leqslant t_1 \leqslant t_2 \leqslant 1$ implies $a(t_1) \leqslant a(t_2)$; and $\mu\big(a(t)\big) = t\mu(a)$ for all $0 \leqslant t \leqslant 1$. A Boolean algebra is said to possess a *μ-resolution* if its unit element has a μ-resolution.

The following important theorem concerning the existence of a μ-resolution for a Boolean algebra is due to Liapounoff [4]. A short proof of this result is contained in [3].

THEOREM 3 (Liapounoff). *If $\mathscr{B}$ is a σ-complete Boolean algebra and μ is a measure which is free of atoms, then there exists a μ-resolution of $\mathscr{B}$.*

We are now in a position to prove the following result:

THEOREM 4. *Let $\mathscr{B}$ be a non-atomic hyperstonian Boolean algebra. Then $\mathscr{B}$ has a continuous resolution.*

Proof. We shall first show that if a non-atomic hyperstonian Boolean algebra has a strictly positive normal state μ, then there exists a continuous resolution of $\mathscr{B}$. To this end, observe that since $\mathscr{B}$ has no atoms, Lemma 2 implies that there are no μ-atoms, and so by Liapounoff's theorem there exists a μ-resolution $\{e(t) : 0 \leqslant t \leqslant 1\}$ of the unit element of $\mathscr{B}$. Then it is easy to see that the strict positivity of μ implies that $\{e(t) : 0 \leqslant t \leqslant 1\}$ is a continuous resolution of $\mathscr{B}$. Indeed, it is only necessary to observe that in this case $a_\tau \uparrow$ and $\mu(a_\tau) \uparrow \mu(a)$ implies $\sup a_\tau = a$ and, similarly, $a_\tau \downarrow$ and $\mu(a_\tau) \downarrow \mu(a)$ implies $a_\tau \downarrow a$.

In the general case $\mathscr{B}$ has sufficiently many normal states but of course none of them may be strictly positive. Let $\{\mu_\tau : \tau \in T\}$ be maximal disjointed system of normal states satisfying $\mu_\tau(1) = 1$ for all $\tau \in T$. Then for every τ we define

$$-a_\tau = \sup\big(a : \mu_\tau(a) = 0\big).$$

Then $\mu_\tau(a_\tau) = 1$ and μ_τ is strictly positive on the ideal generated by a_τ. Furthermore, the maximality of the system $\{\mu_\tau : \tau \in T\}$ implies that $\sup(a_\tau : \tau \in T) = 1$. Thus from what we have just shown it follows that for every τ there exists a continuous resolution

$$\{e_\tau(t) : 0 \leqslant t \leqslant 1\} \text{ of } a_\tau .$$

Then we shall prove that the system

$$\{e(t) = \sup\big(e_\tau(t) : \tau \in T\big),\ 0 \leqslant t \leqslant 1\}$$

is a continuous resolution of $\mathscr{B}$. To this end, we first observe that $e(0) = 0$, and $e(1) = \sup(a_\tau : \tau \in T) = 1$. Let $0 < t_0 \leqslant 1$. Then

$$\sup\big(e(t) : 0 \leqslant t < t_0\big) \geqslant e_\tau(t)$$

for all $\tau \in T$ and for all $0 \leqslant t < t_0$, and so

$$e(t_0) \geqslant \sup\big(e(t) : 0 \leqslant t < t_0\big) \geqslant e_\tau(t_0)$$

for all $\tau \in T$ which implies that

$$e(t_0) = \sup\big(e(t) : 0 \leqslant t < t_0\big).$$

Let $0 \leqslant t_0 < 1$ and let $a = \inf\big(e(t) : t_0 < t \leqslant 1\big)$. Then $a \geqslant e(t_0)$, and since $e(t) \wedge a_\tau = e_\tau(t)$ for all τ and all $0 \leqslant t \leqslant 1$ we obtain $e_\tau(t_0) \leqslant a \wedge a_\tau \leqslant e_\tau(t)$ for all $t_0 < t \leqslant 1$. Thus $e_\tau(t_0) \leqslant a \wedge a_\tau \leqslant e_\tau(t_0)$ implies $e_\tau(t_0) = a \wedge a_\tau$ for all τ. Then, finally, $\sup(a_\tau : \tau \in T) = 1$ implies that

$$e(t_0) = \sup(a \wedge a_\tau : \tau \in T) = a,$$

and the proof is finished.

Using the results of the preceeding section we obtain the following corollaries of Theorem 4.

THEOREM 5. *Every prime ideal in a non-atomic hyperstonian Boolean algebra is σ-incomplete.*

For hyperstonian topological spaces we have the following result:

THEOREM 6. *If X is a compact Hausdorff hyperstonian space without discrete points, then X has no P-points.*

In Lemma 2 we showed that on a non-atomic complete Boolean algebra the normal states are free of atoms. In this direction we have the following result for measures:

THEOREM 7. *If a Boolean algebra has a continuous resolution, then every measure has no atoms. In particular, every measure on a non-atomic hyperstonian Boolean algebra is free of atoms.*

Proof. Let μ be a measure on a Boolean algebra $\mathscr{B}$ possessing a continuous resolution and let $a_0 \in \mathscr{B}$ be a μ-atom. Then the set $I = \{a : \mu(a \wedge a_0) = 0\}$ is a σ-prime ideal which contradicts Theorem 4. Thus μ has no atoms and the proof is finished.

Remark. It seems tempting to conjecture that on every non-atomic hyperstonian Boolean algebra every measure is normal (**P 631**). In this direction some results have been obtained by the present author for the theory of Riesz spaces (see [5] and [6]). These results can immediately be translated into theorems for the theory of states on Boolean algebras when applied to the Riesz space of the finitely valued place function (see [1]) on a Boolean algebra. We shall quote the following results which are related to the conjecture stated above:

(i) *Let $\mathscr{B}$ be any Boolean algebra and let μ be a measure on $\mathscr{B}$. Then there exists a dense ideal $I_\mu \subset \mathscr{B}$ such that μ is normal on I_μ, and equivalently, the null ideal $N_\mu = \{a : \mu(a) = 0\}$ of every singular measure is dense, and equivalently, every strictly positive measure is normal.*

(ii) *The following two statements are equivalent:*

(a) *Every measure defined on the Boolean algebra of all subsets of a non-empty set X for which every finite subset of X is a set of measure zero is identically zero.*

(b) *For all complete Boolean algebras every measure is normal.*

The reader should observe that (a) of (ii) implies that every cardinal is non-measurable. Indeed, the statement (a) if restricted to two-valued measures is precisely the statement: "Every cardinal is non-measurable" as defined in section 1 of this paper. It was shown, however, by Ulam in [9] that if the continuum hypothesis holds, then the following two statements are logically equivalent: (i) Every cardinal is non-measurable. (ii) The statement contained in (a).

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Reçu par la Rédaction le 13. 3. 1967