COLLOQUIUM MATHEMATICUM

VOL. XIX 1968 FASC. 1

ON POWERS OF BASES IN SOME COMPACT ALGEBRAS

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We follow the notation of [6]. In particular, a subset of a general algebra is called a *basis* if it is independent and generates this algebra. Our algebras may have any (infinite) number of finitary operations.

If an algebra has bases of different powers, then these powers are finite and they form an arithmetical progression (see [6] and [1]). Conversely, for every arithmetical progression P of natural numbers there exists an algebra $\mathfrak{A}$ (and even a module) such that n is a power of a basis of $\mathfrak{A}$ if and only if n is in P. This was first proved by Świerczkowski [8], and later, for modules, by Leawitt [4]. For earlier related results see [5] and [1].

It is the purpose of this paper to show that certain algebras do not have bases of different cardinalities. Let us denote by $\mathscr{P}$ this property of algebras. Probably the oldest known result of that kind is that all groups have property $\mathscr{P}$. Jónsson and Tarski have proved in [3] that all free algebras in a class containing a finite algebra, having more than one element, have property $\mathscr{P}$. In particular, finite algebras have property $\mathscr{P}$. In this paper we try to generalize this fact to topological compact algebras and equationally compact algebras.

An algebra $(A, \boldsymbol{F})$ is called *topological* if A is a topological space and all operations $f \in \boldsymbol{F}$ are continuous. $\approx$ denotes homeomorphism of spaces. m and n denote always natural numbers and X^n denotes the Cartesian topological power of the space X. $|X|$ denotes the power of X.

The following Lemma will be applied here:

LEMMA 1. *An algebra has bases of powers m and n $(m > n)$ if and only if it has algebraic operations*

$$\varphi_i(x_1, \ldots, x_n), \quad \psi_j(x_1, \ldots, x_m), \quad i = 1, \ldots, m, \quad j = 1, \ldots, n,$$

satisfying the identities

$$(*) \qquad \varphi_i\big(\psi_1(x_1, \ldots, x_m), \ldots, \psi_n(x_1, \ldots, x_m)\big) = x_i, \quad i = 1, \ldots, m,$$

$$(**) \qquad \psi_j\big(\varphi_1(x_1, \ldots, x_n), \ldots, \varphi_m(x_1, \ldots, x_n)\big) = x_j, \quad j = 1, \ldots, n.$$

For the proof see [1], Theorem 2 or [8], Lemma 1. Our Lemma 1 immediately implies the following

THEOREM 2. *If* $(A, \boldsymbol{F})$ *is a topological algebra with bases of powers* m *and* n, *then* $A^m \approx A^n$.

LEMMA 3. *If* $n \neq m$ *and* A *is a compact topological space satisfying one of the following conditions*:

(i) A *is a metric space and* $0 < \dim A < \infty$ (1),

(ii) A *is a dispersed* (2) *space*,

then A^n *and* A^m *are not homeomorphic.*

Proof. (i) Let A satisfy (i). It is known (see e.g. [2]) that if A and B are compact metric spaces such that $0 < \dim A \leqslant \dim B < \infty$, then $\dim A \times B \geqslant \max(1+\dim A,\ 1+\dim B)$. Thus, for $m > n$, $\dim A^m > \dim A^n$ and, consequently, A^m and A^n are not homeomorphic.

(ii) We may suppose that A is an infinite dispersed space. For $B \subseteq A$ we denote by $c(B)$ the set of all condensation points of B. We put

$$c^{\alpha+1}(B) = c\big(c^{\alpha}(B)\big) \quad \text{and} \quad c^{\lambda} = \bigcap_{\xi<\lambda} c^{\xi}(B)$$

for limit λ. Let $\alpha(k)$ be the first ordinal α such that $c^{\alpha}(A^k) = 0$ (since A is dispersed, such an ordinal exists). We shall show that $m > n$ implies $\alpha(m) > \alpha(n)$.

Let $m > n$. Because A^n is compact and $c^{\xi}(A^n)$, $\big(\xi < \alpha(n)\big)$ is a decreasing family of closed non-empty sets, we have $c^{\lambda}(A^n) \neq 0$ for every limit ordinal $\lambda < \alpha(n)$. Thus, since $c^{\alpha(n)}(A^n) = 0$, there exists a predecessor $\alpha(n)-1$. Let $a \in c^{\alpha(n)-1}(A^n)$. $c(\{a\} \times A^{m-n})$ is non-empty. Since $\{a\} \times A^{m-n} \subseteq c^{\alpha(n)-1}(A^m)$, we have $c^{\alpha(n)}(A^m) \neq 0$ and thus $\alpha(n) > \alpha(m)$.

Hence A^m and A^n are not homeomorphic, q. e. d.

From Theorem 2 and Lemma 3 we obtain immediately

THEOREM 4. *If* $(A, \boldsymbol{F})$ *is a compact topological algebra in which the space* A *satisfies* (i) *or* (ii), *then* $(A, \boldsymbol{F})$ *has the property* $\mathscr{P}$.

THEOREM 5. *If* X *and* Y *are compact* T_1 *spaces and* X *contains a closed subset homeomorphic with* $X \times Y$, *then* $|X| \geqslant |Y|^{\aleph_0}$. *In particular, if* $X^n \approx X^m$, $n \neq m$ *and* X *is a compact* T_1 *space, then* $|X| = |X|^{\aleph_0}$.

Proof. Suppose that for some $n \geqslant 1$ there is a system of closed sets $A_{y_1,\ldots,y_n}$ $(y_k \in Y)$ all homeomorphic with X and such that

$$A_{y_1,\ldots,y_n} \cap A_{y'_1\ldots,y'_n} = 0$$

for $(y_1, \ldots, y_n) \neq (y'_1, \ldots, y'_n)$. The set $A_{y_1,\ldots,y_n}$ being homeomorphic with X there are $|Y|$ disjoint closed subsets of $A_{y_1,\ldots,y_n}$ homeomorphic

(1) We use dim in the sense of [2], p. 24.

(2) That is A has no subset B such that for every open set $G \subseteq A$ intersecting B we have $|B \cap G| = |A|$.

with X. Denote them by $A_{y_1,\ldots,y_n,y_{n+1}} (y_{n+1} \in Y)$. Clearly, all intersections $\bigcap_{n<\omega} A_{y_1,\ldots,y_n}$ are disjoint and non empty, q. e. d.

Let $\mathfrak{m}$ be a cardinal number. As in [8], an algebra $(A, \boldsymbol{F})$ is called *equationally* $\mathfrak{m}$*-compact* if every set of at most $\mathfrak{m}$ algebraic equations with constants in A is solvable whenever every finite subset of this set is solvable. Recall that a topological compact T_2 algebra is equationally $\mathfrak{m}$-compact for every $\mathfrak{m}$ (see [7]).

THEOREM 6. *If* $(A, \boldsymbol{F})$ *is an equationally* $\aleph_0$*-compact algebra in which there exist algebraic operations satisfying identities* $(*)$, *then* $|A| = |A|^{\aleph_0}$.

Proof. We consider the equations

$$(*_*{*})\qquad \begin{aligned} &\psi_1(c_1^1, \ldots, c_{m-n}^1, t_1^1, \ldots, t_n^1) = t_1^0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\psi_n(c_1^1, \ldots, c_{m-n}^1, t_1^1, \ldots, t_n^1) = t_n^0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\psi_1(c_1^p, \ldots, c_{m-n}^p, t_1^p, \ldots, t_n^p) = t_1^{p-1}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\psi_n(c_1^p, \ldots, c_{m-n}^p, t_1^p, \ldots, t_n^p) = t_n^{p-1}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

where c_j^i are constants and t_j^i are unknowns.

Every finite subset of $(*_*{*})$ has a solution. In fact, we can solve it starting from the bottom. Thus $\mathfrak{A}$ being equationally $\aleph_0$-compact, the whole set $(*_*{*})$ has a solution.

Now $(*_*{*})$ makes it possible to define both systems c_k^i and t_j^i $(i = 1, 2, \ldots, k = 1, 2, \ldots, m-n, j = 1, 2, \ldots, n)$ by means of the sequence $t_1^0, \ldots, t_n^0$. Indeed, by $(*)$ we get from $(*_*{*})$ the following equations:

$$t_j^{i+1} = \varphi_{m-n+j}(t_1^i, \ldots, t_n^i) \quad \text{and} \quad c_k^{i+1} = \varphi_k(t_1^i, \ldots, t_n^i).$$

In particular, it is possible to reproduce from $t_1^0, \ldots, t_n^0$ the system of constants c_k^i which were quite arbitrary. Hence we get a mapping of A^n onto $A^{\aleph_0}$ and so $|A| = |A|^{\aleph_0}$, q. e. d.

THEOREM 7. *If* $\mathfrak{A} = (A, \boldsymbol{F})$ *is a compact* T_1*-algebra or an equationally* $\aleph_0$*-compact algebra and* $|A| \neq |A|^{\aleph_0}$ *or* $|\boldsymbol{F}| \neq |\boldsymbol{F}|^{\aleph_0}$, *then* $\mathfrak{A}$ *has property* $\mathscr{P}$. *In particular, both for compact* T_1*-algebras and for equationally* $\aleph_0$*-compact algebras if* $|\boldsymbol{F}| < 2^{\aleph_0}$, *then* $\mathfrak{A}$ *has property* $\mathscr{P}$.

Proof. For $|A| \neq |A|^{\aleph_0}$ the proof follows directly from Theorems 5 and 6.

Now we consider the case $|\boldsymbol{F}| \neq |\boldsymbol{F}|^{\aleph_0}$. Assume that $\mathfrak{A}$ does not have property $\mathscr{P}$. Thus $\mathfrak{A}$ is infinite and there are $n, k > 0$ such that $\mathfrak{A}$ has bases of power $n+ik$ for $i = 0, 1, 2, \ldots$ As in [6], $\boldsymbol{A}^{(n)}$ denotes the set of all algebraic operations of n variables and

$$\boldsymbol{A} = \bigcup_{n<\omega} \boldsymbol{A}^{(n)}.$$

Since $\mathfrak{A}$ has an n-element basis $|\boldsymbol{A}^{(n)}| = |A|$ and by the previous case, it suffices to show that $|\boldsymbol{A}^{(n)}| \neq |\boldsymbol{A}^{(n)}|^{\aleph_0}$.

Since $|\boldsymbol{F}| \neq |\boldsymbol{F}|^{\aleph_0}$, we have $|\boldsymbol{A}| \neq |\boldsymbol{A}|^{\aleph_0}$. Hence it is enough to show $|\boldsymbol{A}^{(n)}| = |\boldsymbol{A}|$. Since $|\boldsymbol{A}^{(n)}| = |\boldsymbol{A}^{(n+ik)}|$ for $i = 0, 1, 2, \ldots$ and $|\boldsymbol{A}^{(n)}| \geqslant \aleph_0$, we have also

$$\Big|\bigcup_{i<\omega}^{\infty} \boldsymbol{A}^{(n+ik)}\Big| = \Big|\bigcup_{i<\omega}^{\infty} \boldsymbol{A}^{(i)}\Big| = |\boldsymbol{A}|,$$

q.e.d.

The following problems related to the subject of this paper are open:

1. Does there exist a topological compact (equationally compact) algebra having bases of different powers? (**P 628**)

2. Let X be a compact space. Does $X^m \approx X^n$ with $m \neq n$ imply $X \approx X^2$? (**P 629**)

3. Let X be a compact connected metric space and $X \approx X^2$. Is then $X \approx X^{\aleph_0}$? (**P 630**)

We due to T. S. Wu the idea of the following remark related to this question. The space $X = (C \times A^{\aleph_0}) \cup_d A^{\aleph_0}$, where C is the Cantor discontinuum, A is any space and $\cup_d$ denotes the disjoint union, satisfies $X^2 \approx X$. But $X^{\aleph_0} \approx X$ fails if A is connected.

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Reçu par la Rédaction le 16. 1. 1967