COLLOQUIUM MATHEMATICUM

VOL. XIX 1968 FASC. 1

TOPOLOGICALLY COMPACT STRUCTURES AND POSITIVE FORMULAS

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0. Introduction. In this paper we give some remarks on positive formulas and topological compact relational structures. Theorem 1.3 says that any structure is upward positively pure in its compactifications. Theorem 2.1 says that for some unary algebras there are compact elementary extensions. Theorem 3.1 says that each positive formula is preserved by inverse limits of systems of topological compact structures. This has been remarked but not published about three years ago by Yu. L. Eršov for the case of finite structures. We also give examples to show the necessity of the conditions in those theorems. This paper is related to [5], where the Čech-Stone compactifications were applied to algebras with unary operations. For other considerations on compact general algebras see also [1].

The terminology and notation of [6] and [7] will be used in this paper. For all topological notions see [2].

A relational structure $\langle A, R_t\rangle_{t\in T}$ will be called *topological* (*completely regular*) [*compact*] if A is a topological Hausdorff space (completely regular) [compact] and each R_t is a closed subset of the topological power A^{r_t}, where r_t is the rank of R_t. The similarity type, that is, the mapping $r\colon T\to \omega\setminus\{0\}$, is often arbitrary but fixed. Sometimes topological algebras are treated as topological relational structures with the n-ary operations converted in the standard way into $(n+1)$-ary relations and, of course, the algebraic operations are supposed to be continuous.

Let $\mathfrak{A} = \langle A, R_t\rangle_{t\in T}$ and $\mathfrak{B} = \langle B, S_t\rangle_{t\in T}$ be topological structures. $\mathfrak{B}$ will be called a *topological extension* of $\mathfrak{A}$ if and only if A is a dense subset of B and $S_t = \overline{R}_t$ for all $t\in T$. $\mathfrak{B}$ is said to be a *topological compactification* of $\mathfrak{A}$ if, moreover, B is a compact space (a topological extension of an algebra need not be an algebra).

1. Topological extensions. Notice the following elementary fact:

PROPOSITION 1.1. *Every topological completely regular structure $\mathfrak{A}$ has a compactification $\mathfrak{B}$ which is maximal, that is, every other compactification of $\mathfrak{A}$ is a continuous homomorphic* (1) *image of $\mathfrak{B}$ by a homomorphism which is the identity on $\mathfrak{A}$.*

Remark. This proposition does not apply to algebras in general (that is, sometimes $\mathfrak{A}$ is but $\mathfrak{B}$ can not be an algebra) except if $\mathfrak{A}$ has only unary operations, since then the Čech-Stone compactification works (see [5]). Moreover, there are algebras which have no equational compactifications (see [6], § 5).

THEOREM 1.2. *Let $\mathfrak{A} = \langle A, R_t\rangle_{t\in T}$ be a topological compact structure and $\varphi(x_0, \ldots, x_{n-1})$ a positive formula. Then the set*

$$\{\langle a_0, \ldots, a_{n-1}\rangle : \mathfrak{A} \models \varphi[a_0, \ldots, a_{n-1}]\}$$

is a closed subset of A^n.

Proof. We proceed by induction with respect to φ and apply the well known facts that the logical operations $\vee$, $\exists$, $\wedge$ and $\forall$ correspond to union, projection and intersections respectively.

Let us recall that $\mathfrak{A}$ is upward positively pure in $\mathfrak{B}$ if and only if $\mathfrak{A} \subseteq \mathfrak{B}$ and each positive formula with constants in $\mathfrak{A}$ which is satisfiable in $\mathfrak{A}$ is also satisfiable in $\mathfrak{B}$. (For more informations on this notion see [7].)

THEOREM 1.3. *If $\mathfrak{A}$ is a topological structure and $\mathfrak{B}$ is any topological compactification of $\mathfrak{A}$, then $\mathfrak{A}$ is upward positively pure in $\mathfrak{B}$.*

Proof. We proceed by induction on φ. For atomic formulas or for formulas of the form $\varphi \wedge \psi$, $\varphi \vee \psi$ or $\exists x\varphi$, where φ and ψ already have the required property, the conclusion is visible. We have still to consider $\chi = \forall x_0 \varphi(x_0, \ldots, x_n)$. If χ is satisfiable in $\mathfrak{A}$, that is, there are $a_1, \ldots, a_n \in A$ such that

$$\mathfrak{A} \models \chi[a_1, \ldots, a_n],$$

then we have $\mathfrak{A} \models \varphi[a_0, a_1, \ldots, a_n]$ for each $a_0 \in A$ and by the inductive supposition we also have $\mathfrak{B} \models \varphi[a_0, a_1, \ldots, a_n]$. Hence the set

$$\{b \in B : \mathfrak{B} \models \varphi[b, a_1, \ldots, a_n]\}$$

is dense in B, and by Theorem 1.2, it is closed and thus equals B, q. e. d.

Next, we add two examples which show that compactness of $\mathfrak{B}$ in 1.3 and 1.2 is essential and $\mathfrak{A}$ needs not be downward positively pure in $\mathfrak{B}$.

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(1) We recall that a *homomorphism* $f: \langle A, R_t\rangle_{t\in T} \to \langle B, S_t\rangle_{t\in T}$ is a mapping $f: A \to B$ such that, for every $t \in T$ and $a_1, \ldots, a_{r_t} \in A$, if $\langle a_1, \ldots, a_{r_t}\rangle \in R_t$, then $\langle f(a_1), \ldots, \ldots, f(a_{r_t})\rangle \in S_t$.

Example 1.4. Let $\mathfrak{A}$ be the multiplicative group of real numbers without zero, and $\mathfrak{B}$ the multiplicative semigroup of all real numbers. Then $\mathfrak{B}$ is a topological extension of $\mathfrak{A}$, but $\mathfrak{A}$ is not upward positively pure in $\mathfrak{B}$ (e.g. $\forall x \exists y xy = 1$).

Example 1.5. Let $\mathfrak{A} = \langle N, G, F\rangle$, where F and G are binary relations on the set N of all natural numbers with the discrete topology defined by $F(x, y) \Leftrightarrow f(x) = y$ and $G(x, y) \Leftrightarrow g(x) = y$, where f and g are two functions such that

$$(f, g): N \xrightarrow{\text{onto}} N \times N \setminus \{(x, x): x \in N\}.$$

Then $\mathfrak{A}$ has no compactification in which it is downward positively pure (see [7], Example 21, due to C. Ryll-Nardzewski).

2. The Čech-Stone compactifications. In this section we define a certain class of discrete algebras for which the Čech-Stone compactifications are elementary extensions.

Let $\boldsymbol{K}$ be the class of all algebras $\mathfrak{A} = \langle A, f\rangle$ such that f is unary operation and none of the functions $f, f^2, \ldots, f^n, \ldots$, where $f^{n+1}(x) = f(f^n(x))$, has a fixed point.

THEOREM 2.1. *For every $\mathfrak{A} \in \boldsymbol{K}$ the Čech-Stone compactification of $\mathfrak{A}$ is an elementary extension of $\mathfrak{A}$.*

For the proof of this theorem we need some lemmas.

LEMMA 2.2. (C. Ryll-Nardzewski). *For any $\mathfrak{A} \in \boldsymbol{K}$ there are sets A_0, A_1, A_2 such that $A_0 \cup A_1 \cup A_2 = A$, $A_i \cap A_j = 0$ for $i \neq j$ and $f(A_i) \cap A = 0$ for $i, j < 3$.*

The proof is elementary.

Let $\boldsymbol{F}$ be the family of all sets of the form

$$\{x \in (\beta A)^\omega: x_{m_0} = f^{n_0}(a), \ldots, x_{m_{k-1}} = f^{n_{k-1}}(a) \quad \text{where}$$
$$a \in \beta A \setminus \{a_1, \ldots, a_j\},\ a_1, \ldots, a_j \in A,\ j < \omega\}$$

and let D be the family of all sets of the form

(1) $$D = \bigcup_{k=1}^{n} (F_k \setminus \bigcup_{l=1}^{r_k} F_{kl}), \quad \text{where} \quad F_k, F_{kl} \in \boldsymbol{F}.$$

LEMMA 2.3. *The family $\boldsymbol{D}$ is closed under finite unions, complements and cylindrifications $\mathscr{C}_i$ $(i < \omega)$, defined as follows:*

$$\mathscr{C}_i(D) = \{x \in (\beta A)^\omega: \textit{there is an } y \in \beta A \textit{ such that } x(i|y) \in D\},$$

where $x(i|y)$ denotes the sequence obtained from x by substituting y for x_i on the i-th place.

Proof. For each $0 < n < \omega$, f^n has no fixed points, hence, using Lemma 2.2, we have three sets A_0, A_1 and A_2 such that $\beta A = \bar{A}_0 \cup \bar{A}_1 \cup \bar{A}_2$ and $\bar{A}_i \cap \bar{A}_j = 0$ for $i \neq j$, $i, j < 3$. Let $b \in \beta A$. Then $b \in \bar{A}_i$ for some $i < 3$; say $b \in \bar{A}_0$. Then $f^n(b) \in f^n(\bar{A}_0) \subseteq \bar{A}_1 \cup \bar{A}_2$, but $\bar{A}_0 \cap (\bar{A}_1 \cup \bar{A}_2) = 0$, thus $f^n(b) \neq b$ for all $b \in \beta A$. From this it immediately follows that $\boldsymbol{F}$ is closed under finite non-void intersections.

In the representation of the set $D \in \boldsymbol{D}$ of the form (1) we can assume that

$$F_k \supseteq \bigcup_{l=1}^{r_k} F_{kl}.$$

Whence it is almost obvious that $\boldsymbol{D}$ is closed under finite unions and since $\boldsymbol{F}$ is closed under finite non-void intersections, $\boldsymbol{D}$ is also closed under complements. To show that $\boldsymbol{D}$ is closed under operations $\mathscr{C}_i$ $(i < \omega)$ let us observe that

$$\mathscr{C}_i(F \setminus \bigcup_{n=1}^{r} F_n) = \mathscr{C}_i(F) \setminus \bigcup_{n=1}^{r} \mathscr{C}_i(F_n \cap E_n),$$

where

$$E_n = \begin{cases} 0 & \text{if} \quad \mathscr{C}_i(F_n) = F_n \text{ and } \mathscr{C}_i(F) = F, \\ (\beta A)^\omega & \text{in} \quad \text{the other cases.} \end{cases}$$

Whence, using the fact that $\mathscr{C}_i(F) \in \boldsymbol{F}$ for $F \in \boldsymbol{F}$, we see that $\boldsymbol{D}$ is closed under operations $\mathscr{C}_i$ $(i < \omega)$, q. e. d.

For all $j < \omega$ we put

$$X_j = \{x \in (\beta A)^\omega : x_k = a_k \text{ for all } k \neq j, k < \omega\},$$

where $a_k \in A$.

LEMMA 2.4. *If $X_j \cap D \neq 0$ for some $D \in \boldsymbol{D}$, then $X_j \cap D \cap A^\omega \neq 0$.*

Proof. For every $F \in \boldsymbol{F}$ the projection $p_j(F \cap X_j)$ contains either all points from βA but finite number from A or there is only one point belonging to A. Hence if $D = F \setminus \bigcup_{k=1}^{n} F_k$, then the conclusion of Lemma 2.4 follows from the fact that

$$p_j(D \cap X_j) = p_j(F \cap X_j) \setminus \bigcup_{k=1}^{n} p_j(F_k \cap X_j).$$

Proof of Theorem 2.1. Let $\mathfrak{A} \in \boldsymbol{K}$. Then $\mathfrak{A}$ is infinite. Let us observe that the sets defined in $\beta A = \langle \beta A, f^* \rangle$ by atomic formulas belong to $\boldsymbol{F}$. Since $\boldsymbol{D}$ is closed under finite unions, complements and cylindrifications $\mathscr{C}_i$ $(i < \omega)$, the sets defined by elementary formulas in βA belong to $\boldsymbol{D}$.

Thus, in view of a theorem of Tarski and Vaught ([4], Theorem 1.10), Theorem 2.1 follows from Lemma 2.4 (2).

3. Inverse limits of topological relational structures. Now we are going to prove a generalization of the result of Eršov mentioned in the introduction.

THEOREM 3.1. *If* $\langle \mathfrak{A}_i, \pi_{ij}, I\rangle$ *is an inverse system of topological compact structures such that* π_{ij} *are continuous homomorphisms onto, and* $\mathfrak{B}$ *is the inverse limit of this system, then for each positive formula* φ *and* $b_0, \ldots, b_{n-1} \in B$ *we have*

$$\mathfrak{B} \models \varphi[b_0, \ldots, b_{n-1}]$$

if and only if $\mathfrak{A}_i \models \varphi[\pi_i(b_0), \ldots, \pi_i(b_{n-1})]$ *for each* $i \in I$, *where* π_i *is the natural projection of* $\mathfrak{B}$ *onto* $\mathfrak{A}_i$.

In particular, positive sentences true in all $\mathfrak{A}_i$ *are true in* $\mathfrak{B}$.

Proof. If $\mathfrak{B} \models \varphi[b_0, \ldots, b_{n-1}]$, then, by a theorem of Marczewski [3], $\mathfrak{A}_i \models \varphi[\pi_i(b_0), \ldots, \pi_i(b_{n-1})]$ for each positive formula and all $i \in I$.

We prove the converse by induction. The thesis is visible if φ is an atomic formula, and it is almost obvious if φ is of one of the forms $\psi_1 \wedge \psi_2$, $\psi_1 \vee \psi_2$ and $\forall x_n \psi$.

We have still to consider the formula

$$\varphi = \exists x_n \psi(x_0, \ldots, x_{n-1}, x_n),$$

where φ has the required property. Let us suppose that we have $\mathfrak{A}_i \models \varphi[\pi_i(b_0), \ldots, \pi_i(b_{n-1})]$ for some $b_0, \ldots, b_{n-1} \in B$ and all $i \in I$. Let us denote by F_i the set

$$F_i = \{a \in A_i \colon \mathfrak{A}_i \models \psi[\pi_i(b_0), \ldots, \pi_i(b_{n-1}), a]\}.$$

Since $\mathfrak{A}_i \models \varphi[\pi_i(b_0), \ldots, \pi_i(b_{n-1})]$, we have $F_i \neq 0$, and, by Theorem 1.2, F_i is a compact subset of A_i. Moreover, if $i \leqslant j$, then π_{ij} induces a mapping $\varrho_{ij}\colon F_j \xrightarrow{\text{into}} F_i$. Thus we obtain an inverse system of topological compact spaces $\langle F_i, \varrho_{ij}, I\rangle$. Let F be the inverse limit of the system $\langle F_i, \varrho_{ij}, I\rangle$. It is a subset of B. Since each F_i is non-void and compact, F is so as well. Let $b \in F$. Then we have

$$\mathfrak{A}_i \models \psi[\pi_i(b_0), \ldots, \pi_i(b_{n-1}), \pi_i(b)]$$

for all $i \in I$. So, by the inductive supposition, $\mathfrak{B} \models \psi[b_0, \ldots, b_{n-1}, b]$, whence $\mathfrak{B} \models \exists x_n \psi$, q. e. d.

(2) A structure $\mathfrak{B}_1$, which is an extension of $\mathfrak{B}_2$, is an *elementary extension* of $\mathfrak{B}_2$ if and only if the existence of an $a \in B_1$ such that $\mathfrak{B}_1 \models \varphi[a_0, \ldots, a_{n-1}, a]$ implies the existence of a $b \in B_2$ such that $\mathfrak{B}_1 \models \varphi[a_0, \ldots, a_{n-1}, b]$ for every natural number n, for every formula φ with $n+1$ free variables and for every $a_0, \ldots, a_{n-1} \in B_2$.

The following example shows that the assumption of compactness of A_i in Theorem 3.1 is essential.

Example 3.2. Let $\mathfrak{A}_n = \langle A_n, f_n \rangle$, $n < \omega$, where

$$A_n = \{\langle n, k, r\rangle : k < n, r < \omega\} \cup \{\langle n, s, 0\rangle : s \geqslant n\},$$

and f_n is a unary operation over A_n defined by

$$f_n(\langle n, k, r\rangle) = \begin{cases} \langle n, k, r+1\rangle & \text{if} \quad k < n, \\ \langle n, k, r\rangle & \text{if} \quad k \geqslant n. \end{cases}$$

Let $\pi_{n+1,n}$ be a mapping defined by the following conditions:

1° $\pi_{n+1,n}\colon A_{n+1} \xrightarrow{\text{onto}} A_n$;

2° $\pi_{n+1,n}(\langle n+1, k, r\rangle) = \begin{cases} \langle n, k, r\rangle & \text{if} \quad k < n, \\ \langle n, k, 0\rangle & \text{if} \quad k \geqslant n. \end{cases}$

Let $\mathfrak{B}$ be the inverse limit of the system $\langle \mathfrak{A}_n, \pi_{ji}, i < j < \omega\rangle$. It is visible that $\mathfrak{A}_n \models \exists x[f(x) = x]$ for each $n < \omega$, but $\mathfrak{B} \models \forall x[f(x) \neq x]$.

Let us remark that in this example each $\mathfrak{A}_n$ is atomic compact (see [6]) and that we do not know any natural purely algebraic assumption which could replace the topological compactness in Theorem 3.1.

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Reçu par la Rédaction le 14. 2. 1967