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ON THE NUMBER OF INDEPENDENT ELEMENTS IN FINITE ABSTRACT ALGEBRAS WITH TWO BINARY SYMMETRICAL OPERATIONS

BY

J. PŁONKA (WROCŁAW)

0. In this paper we consider abstract algebras which have at least two binary symmetrical algebraic operations. Commutative rings and lattices may serve as examples of such algebras. Our main result is contained in section 2, where we evaluate the minimal number of elements which an algebra of our class with a given number of independent elements may possess. Problems of this kind were suggested to me by Professor E. Marczewski and were solved in previous papers [2] and [4] for unary algebras and for algebras having one binary operation.

We adopt the terminology of [1] where, in particular, the definition of independence may be found. For shortness, we shall say *constant* instead of *algebraic constant*, and *essentially n-ary operation* instead of *n-ary operation depending on each its variable.*

1. We start with a constructive description of some simple classes of abstract algebras defined axiomatically, which we shall use in the sequel.

Let $\mathfrak{C} = (X; +, \cdot, 0)$ be an algebra such that 0 is an algebraic constant and the fundamental operations satisfy the equations

(1) $$x+y = y+x,$$

(2) $$x\cdot y = y\cdot x,$$

(3) $$(x+y)+z = (x\cdot y)\cdot z = (x+y)\cdot z = (x\cdot y)+z = x+x = x\cdot x = 0.$$

(i) *Every algebra $\mathfrak{C}$ is of the following form: X is the union of two disjoint sets A and B, there exist two mappings f and g each of them mapping the set $A\times A\setminus\{\langle x, x\rangle : x\in A\}$ into the set B and such that $f(\langle x,y\rangle)=f(\langle y,x\rangle)$, $g(\langle x, y\rangle) = g(\langle y, x\rangle)$ and $0\in B$. The algebraic operations $+$ and $\cdot$ are* *defined by*

$$x+y = \begin{cases} f(\langle x, y\rangle) & \text{if} \quad x \neq y,\, x, y \in A, \\ 0 & \text{otherwise}, \end{cases}$$

$$x \cdot y = \begin{cases} g(\langle x, y\rangle) & \text{if} \quad x \neq y,\, x, y \in A, \\ 0 & \text{otherwise}. \end{cases}$$

Conversely, every algebra constructed in this way satisfies equations (1), (2) *and* (3).

Proof. Let B be the set of all elements x such that $x+y = x \cdot y = 0$ holds for every y, and let A be the complement of B. Now it is clear how to define the mappings f and g to get the wanted result. The converse part can be checked without any difficulty.

(ii) *An algebra $\mathfrak{C}$ is a free algebra with n free generators in the equational class defined by* (1), (2) *and* (3) *if and only if the set A occurring in* (i) *has n elements, f and g are both one-to-one mappings and the images of f and g are disjoint and exhaust the set $B \setminus \{0\}$.*

The easy proof may be left to the reader.

We shall denote by $\mathfrak{C}^{(n)}$ the free algebra described in (ii).

Now let $\mathfrak{A} = (X; +, \cdot)$ be an algebra whose fundamental operations satisfy equations (1) and (2) and also the following:

(4) $$(x+y)+z = x+(y+z),$$

(5) $$(x \cdot y) \cdot z = x(y \cdot z),$$

(6) $$x+y+z = x \cdot y \cdot z = (x+y) \cdot z = (x \cdot y)+z,$$

(7) $$x+x = x \cdot x,$$

(8) $$x \cdot x+y = x \cdot x \cdot y = x+y.$$

(Let us note that, as observed by K. Urbanik, the algebra $\mathfrak{A}$ can be described equivalently as follows: let $x \cdot y$ be a commutative and associative operation defined on X, satisfying the conditions $x^2yz = xyz$ and $x^3 = x^2$. Define the second binary operation by $x+y = x^2 \cdot y^2$. It can be checked that the two descriptions of $\mathfrak{A}$ are equivalent.)

(iii) *Every such algebra $\mathfrak{A}$ is of the following form: the carrier X is the union of three disjoint sets A, B and S, the set S is a carrier of a subalgebra of $\mathfrak{A}$, and the algebra $(S; +)$ is a semilattice (i.e. an idempotent commutative semigroup). Moreover, there exists a homomorphism f of the algebra $(X; +)$ into $(S; +)$ which is a retraction (i.e. $ff = f$), and equation $x+y = f(x)+f(y)$ holds for every pair of elements x, y in X. Finally, there is* *a mapping g, defined on the set of all two-element subsets of $\mathfrak{A}$ with values in the set $B \cup S$, such that $f(g(\{a, b\})) = a+b$, and we have*

$$x \cdot y = \begin{cases} g(x, y) & \text{if } x, y \in A \text{ and } x \neq y, \\ f(x) & \text{if } x, y \in A \text{ and } x = y, \\ f(x)+f(y) & \text{in the remaining cases.} \end{cases}$$

Conversely, every algebra constructed in this way satisfies equations (1), (2) *and* (4)-(8).

Proof. Define, for $x \in X, f(x) = x+x$. Using (1), (4) and (8) we get $f(x+y) = x+x+y+y = f(x)+f(y)$, and also $f(x)+f(y) = x+x+y+y = x \cdot x+y+y = x+y+y = x+y$. Moreover, $f(f(x)) = f(x+x) = x+x = f(x)$. Put $S = f(X)$. For $x \in S$ we have $x+x = f(x) = x$ which shows that $(S; +)$ is a semilattice. Observe also that for $x \in S$ we have $x \cdot y = f(x) \cdot y = x \cdot x \cdot y = x+y$, whence $(S; +, \cdot)$ is a subalgebra of $\mathfrak{A}$. Let B be the set of all such elements $x \notin S$ for which the equation $x \cdot y = x+y$ holds for every $y \in X$, and let $A = X \setminus (B \cup S)$. Define $g(\{x, y\})$ (for $x, y \in A, x \neq y$) by $g(\{x, y\}) = x \cdot y$. It remains to show that $f(x \cdot y) = x+y$, but this results from $f(x \cdot y) = x^2 y^2 = x^2+y^2 = x+y$.

Now we prove the converse. The equations (1), (2) (4) and (7) are evident. Observe that every element of the form $x \cdot y$ belongs to $B \cup S$ and $f(x \cdot y) = f(x)+f(y)$, whence

$$(x \cdot y) \cdot z = f(x \cdot y)+f(z) = f(x)+f(y)+f(z) = f(x)+f(y, z) = x \cdot (y \cdot z),$$

thus proving (5). Similarly, every element of the form $x+y$ belongs also to $B \cup S$, and for every $x \in B \cup S$ and $y \in X$ we have $x \cdot y = x+y$, whence $(x+y) \cdot z = x+y+z$ and

$$x \cdot y \cdot z = (x \cdot y)+z = f(x \cdot y)+f(z) = f(x)+f(y)+f(z) = x+y+z,$$

which proves (6). To prove (8) observe that $x^2 = f(x) \in S$ for all $x \in X$, and so

$$\begin{aligned} x^2 \cdot y = x^2+y = f(x)+y &= f(f(x)+y) \\ &= f(f(x))+f(y) = f(x)+f(y) = x+y \end{aligned}$$

as needed.

(iv) *An algebra $\mathfrak{A}$ is a free algebra with n free generators in the equational class defined by* (1), (2), (4)-(8) *if and only if the set A occurring in* (iii) *has n elements, S is a free semilattice with n free generators, f is a one-to-one mapping of $A \cup B$ onto S which maps A onto the set of free generators of S, and g is a one-to-one mapping onto the set B.*

This follows from (iii).

We shall denote by $\mathfrak{A}_n$ the free algebra described in (iv).

Let $\mathfrak{W} = (X; +, \cdot)$ be an algebra whose fundamental operations satisfy equations (1), (2), (4), (5) and, additionally, also the following:

$$x+x = x, \tag{9}$$

$$x\cdot y+z = x\cdot y\cdot z = (x+y)\cdot z. \tag{10}$$

(v) *Every such algebra* $\mathfrak{W}$ *is of the following form: the algebra* $(X; +)$ *is a semilattice and there exists an endomorphism of this semilattice, say* $r(x)$, *such that* $r(x+y) = r(x)+y$ *and* $x\cdot y = r(x)+r(y)$. *Moreover*, $r\big(r(x)\big) = r(x)$.

Conversely, every algebra of this form satisfies (1), (2), (4), (5), (9), (10).

Proof. Clearly $(X; +)$ is a semilattice and the endomorphism $r(x) = x\cdot x$ satisfies the assertion.

To prove the converse observe that (1), (2), (4) and (9) are easily deduced. Moreover,

$$\begin{aligned}(x\cdot y)\cdot z = r(x\cdot y)+r(z) &= r\big(r(x)+r(y)\big)+r(z)\\ &= r\big(r(x)\big)+r\big(r(y)\big)+r(z) = r(x)+r(y)+r(z).\end{aligned}$$

Similarly, $r(x)+r(y)+r(z) = x\cdot(y\cdot z)$ and so (5) is proved. To prove (10) observe that

$$(x+y)\cdot z = r(x+y)+r(z) = r(x)+r(y)+r(z),$$

$$x\cdot y+z = r(x)+r(y)+z = r(x)+r(y)+r(z) = x\cdot y\cdot z.$$

As an example, consider the semilattice $(a, b, c; +)$, where $a+b = b$, $a+c = c$ and $b+c = c$. Define the retraction r by $r(a) = r(b) = b$ and $r(c) = c$. Clearly, r satisfies $r(x+y) = r(x)+y$, whence, putting $x\cdot y = r(x)+r(y)$, we get an example of an algebra satisfying (1), (2), (4), (5), (9) and (10).

(vi) *Let* $\mathfrak{S} = (X; +)$ *be a semilattice. The mapping* r *satisfies the equation* $r(x+y) = r(x)+y$ *if and only if the following three conditions are satisfied*: r *is a retraction*;

$$x \leqslant r(x) \quad (x\in X); \tag{11}$$

and for every triple $a < b < c$ *of elements of* X *either* $r(a) \neq a$ *or* $r(b) \neq c$ *holds.*

Proof. First assume that $r(x+y) = r(x)+y$ holds identically.

We have

$$\begin{aligned}r\big(r(x)\big) = r\big(r(x+x)\big) &= r\big(r(x)+x\big)\\ &= r\big(x+r(x)\big) = r(x)+r(x) = r(x),\end{aligned}$$

$$\begin{aligned}r(x+y) = r\big(r(x+y)\big) &= r\big(r(x)+y\big)\\ &= r\big(y+r(x)\big) = r(y)+r(x) = r(x)+r(y).\end{aligned}$$

Condition (11) immediately follows (simply put $x = y$). If for some triple $a < b < c$, $r(a) = a$ and $r(b) = c$, then we would have $r(a)+b = b$, $r(a)+r(b) = c > b$, a contradiction.

Now assume that r satisfies the three conditions stated above. We have $r(x) \leqslant r(x)+r(y)$ and so, by (11), $y \leqslant r(y) \leqslant r(x)+r(y)$. Consequently $r(x)+y \leqslant r(x)+r(y)$. Let now x and y be such elements of X for which $r(x)+y < r(x)+r(y)$. Obviously $r\big(r(x)+y\big) = r\big(r(x)\big)+r(y) = r(x)+r(y) \neq r(x)+y$.

The equation $r(x)+y = r(x)$ cannot occur as it would imply $r\big(r(x)+y\big) = r\big(r(x)\big) = r(x) = r(x)+y$ against the choice of x and y. Thus $r(x) < r(x)+y < r(x)+r(y)$. But now, putting $r(x) = a$, $r(x)+y = b$ and $r(x)+ +r(y) = c$, we obtain a situation which is excluded by our assumptions.

(vii) *Let $\mathfrak{S} = (X; +)$ be a semilattice and let $r(x)$ be a retraction satisfying $r(x+y) = r(x)+y$. If $r(a) \neq a$, then for every $b < a$ either $r(b) = r(a)$ or $r(b)$ is incomparable with a.*

Proof. Let $b < y$ and $r(a) \neq a$. The inequality $r(b) < r(a)$ is impossible by (v). If $r(b) > a$, then $r(b) \leqslant r(a)$, hence $r(b) \neq r(a)$ implies $a < r(b) < r(a)$ and $r(b)+a = r(b) < r(b)+r(a) = r(a) \neq r(b)$, a contradiction.

(viii) *If a semilattice $\mathfrak{S}$ is linearly ordered, say $a_1 < a_2 < \ldots$, and $r(x)$ is a retraction satisfying $r(x+y) = r(x)+y$ and such that for some j we have $r(a_j) = a_k$, then $r(a_i) = a_i$ for $i \geqslant k$ and $r(a_i) = a_k$ for $i < k$.*

This follows easily from (vi) and (vii).

2. Let $a(\mathfrak{A})$ denote the number of elements of an algebra $\mathfrak{A}$, and let $\iota(\mathfrak{A})$ denote the maximal cardinality of an independent set in $\mathfrak{A}$. If $\boldsymbol{K}$ is a given class of algebras, then we define

$$p(n, \boldsymbol{K}) = \min\{a(\mathfrak{A}) : \mathfrak{A} \in \boldsymbol{K} \text{ and } \iota(\mathfrak{A}) = n\}.$$

By $\boldsymbol{K}_C^2$ we shall denote the class of all algebras which have constants and at least two different symmetrical essentially binary algebraic operations. We shall use the inequality (see [2])

$$a(\mathfrak{A}) \geqslant \sum_{j=0}^{n} \binom{n}{j} \omega_j \tag{12}$$

which estimates the cardinality of an algebra $\mathfrak{A}$ containing an independent set of n elements and having ω_j essentially j-ary algebraic operations. (Note that $\omega_1 \geqslant 1$, as the operation $e_1^1(x) = x$ is an algebraic one by definition).

We have the following

THEOREM I. *For* $n \geqslant 1$ *we have* $p(n, \boldsymbol{K}_c^2) = n^2+1$.

Proof. The inequality $p(n, \boldsymbol{K}_c^2) \geqslant n^2+1$ is a consequence of (12). To get the reverse inequality consider for $n = 1$ the 2-element lattice with one constant ($=1$), which has one independent element, and for $n \geqslant 2$ consider the algebra $\mathfrak{C}^{(n)}$ as described in (ii). One has only to observe that in $\mathfrak{C}^{(n)}$ the only operations which depend on all their variables are $x+y$, $x\cdot y$, x and the constant operation 0.

Now we shall consider the class $\boldsymbol{K}_w^2$ of algebras, consisting of all algebras without constants with at least two different symmetrical binary essentially algebraic operations. We shall prove the following

THEOREM II. *For* $n \geqslant 3$ *we have* $p(n, \boldsymbol{K}_w^2) = 2^n-1+\binom{n}{2}+n$. *Moreover* $p(1, \boldsymbol{K}_w^2) = 2$ *and* $p(2, \boldsymbol{K}_w^2) = 4$.

To the proof of this theorem we shall need several lemmas. Let $\mathfrak{A} \in \boldsymbol{K}_w^2$ and let $x+y$ and $x\cdot y$ be two different symmetrical essentially binary algebraic operations in $\mathfrak{A}$. Let us introduce the following notations:

$$f_{11}(x, y, z) = (x+y)+z, \quad f_{12}(x, y, z) = (x+z)+y,$$
$$f_{13}(x, y, z) = (y+z)+x, \quad f_{21}(x, y, z) = (x\cdot y)\cdot z,$$
$$f_{22}(x, y, z) = (x\cdot z)\cdot y, \quad f_{23}(x, y, z) = (y\cdot z)\cdot x,$$
$$f_{31}(x, y, z) = (x+y)\cdot z, \quad f_{32}(x, y, z) = (x+z)\cdot y,$$
$$f_{33}(x, y, z) = (y+z)\cdot x, \quad f_{41}(x, y, z) = (x\cdot y)+z,$$
$$f_{42}(x, y, z) = (x\cdot z)+y, \quad f_{43}(x, y, z) = (y\cdot z)+x.$$

LEMMA 1. *If the operations* $x+y$ *and* $x\cdot y$ *are both idempotent, then every operation* $f_{ik}(x, y, z)$, $i, k = 1, 2, 3$, *is essentially ternary.*

Proof. Consider, for example, the operation $f_{31}(x, y, z)$. As $x+y = y+x$, it depends on x if and only if it depends on y, but putting $x = y$ we get xz, which depends on x and z, hence f_{31} depends on all its variables. The proof for other operations f_{ik} is similar.

LEMMA 2. *If the operations* $x+y$ *and* $x\cdot y$ *are both idempotent and* $i \neq j$, *then the operations* f_{ik} *and* f_{jl} *are distinct.*

Proof. If, e.g., $f_{11} = f_{21}$ or $f_{11} = f_{31}$, then putting $x = y$ we get $x+z = x\cdot z$, a contradiction. If $f_{11} = f_{41}$, then putting first $z = x+y$ and then $z = x\cdot y$ we also get $x+y = xy$, a contradiction. If $f_{11} = f_{22}$, then

$$(x+y)+z = (x\cdot z)\cdot y = (z\cdot x)\cdot y = (z+y)+x$$
$$= (y+z)+x = (y\cdot x)\cdot z = (x\cdot y)\cdot z,$$

whence $f_{11} = f_{21}$ which is impossible as we have already shown. Other cases can be dealt with in the same way.

LEMMA 3. *If the operation $x+y$ is idempotent and associative, then f_{31} cannot be symmetrical. Similarly, if $x\cdot y$ is idempotent and associative, then f_{41} cannot be symmetrical.*

Proof. Otherwise we would have

$$x+y = (x+y)\cdot(x+y) = \big(x+(x+y)\big)\cdot y$$
$$= (x+y)\cdot y = (y+x)\cdot y = (y+y)\cdot x = y\cdot x,$$

which is impossible. The second part of the lemma obtains by interchanging $+$ and $\cdot$ in the last chain of equalities.

From lemmas 1-3 we obtain immediately

LEMMA 4. *If $x+y$ and $x\cdot y$ are idempotent, then the algebra $\mathfrak{A}$ has at least 8 distinct essentially ternary operations.*

Proof. In fact, if $x+y$ and $x\cdot y$ are both associative, then $f_{11}, f_{21}, f_{3k}, f_{4k}$ $(k = 1, 2, 3)$ are essentially ternary and distinct; if $x+y$ is associative but $x\cdot y$ is not, then $f_{11}, f_{2k}, f_{3k}, f_{41}$ $(k = 1, 2, 3)$ are such; if $x+y$ is not associative but $x\cdot y$ is associative, then $f_{1k}, f_{21}, f_{31}, f_{4k}$ $(k = 1, 2, 3)$ are such; and, finally, if neither $x+y$ nor $x\cdot y$ are associative, then f_{1k}, f_{2k}, f_{31} and f_{41} $(k = 1, 2, 3)$ are such.

Proof of theorem II. Obviously $p(1, \boldsymbol{K}_w^2) \geqslant 2$ and the two-element lattice realizes the equality. The inequality $p(2, \boldsymbol{K}_w^2) \geqslant 4$ follows from (12) and equality is realized by the lattice consisting of 4 elements $0, 1, a, b$ with $a+b = 1, a\cdot b = 0$.

Now let $n \geqslant 3$. From Theorem 1 of [4] it follows that $\mathfrak{A}$ has essentially j-ary operations for $j = 3, 4, \ldots, n$, and so $\omega_j \geqslant 1$ $(j = 3, 4, \ldots, n)$ and $\omega_2 \geqslant 2$. If $x+y$ and $x\cdot y$ are not both idempotent, then $\omega_1 \geqslant 2$ and, by (12), we get $a(\mathfrak{A}) \geqslant 2^n - 1 + \binom{n}{2} + n$. If both $x+y$ and $x\cdot y$ are idempotent, then the above inequality follows from lemma 4 and (12). In this case the algebra which gives the least value for a is the algebra $\mathfrak{A}_n$ described in (iv). It is an easy exercise to show that for this algebra we have $\omega_1 = \omega_2 = 2$, and $\omega_3 = \ldots = \omega_n = 1$. In fact, the only algebraic operations in $\mathfrak{A}_n$ are the following: $x, x+y, x\cdot y, x+x$ and $x_1+\ldots+x_k (k \geqslant 3)$.

Now let $\boldsymbol{K}^{(2)}$ be the class of all algebras containing two distinct symmetrical non-constant binary operations, one of them being idempotent. We prove the following

THEOREM III. *For $n \geqslant 3$ we have $p(n, \boldsymbol{K}^{(2)}) = 2(2^n - 1)$. Moreover, $p(1, \boldsymbol{K}^{(2)}) = 2$ and $p(2, \boldsymbol{K}^{(2)}) = 4$.*

We shall assume that $x+y$ is idempotent. First we prove that in every algebra from the class $\boldsymbol{K}^{(2)}$ there are at least two essentially n-ary algebraic operations for $n = 2, 3, \ldots$ Observe that $x+y$ and $x\cdot y$ depend on both variables, and consider (for $n = 3, 4, \ldots$) the equations

$$f_n(x_1, \ldots, x_n) = \Big(\big(\ldots(x_1+x_2)+x_3\big)+\ldots+x_{n-1}\Big)+x_n$$

and

$$g_n(x_1, \ldots, x_n) = f_{n-1}(x_1, \ldots, x_{n-1})\cdot x_n.$$

The operation f_n is essentially n-ary by a result of Marczewski [3]. In a similar way, we prove that the operation g_n is essentially n-ary. Let

$$s_2(x_1, x_2) = x_1+x_2$$

and

$$s_{2^n}(x_1, \ldots, x_{2^n}) = s_{2^{n-1}}(x_1, \ldots, x_{2^{n-1}})+s_{2^{n-1}}(x_{2^{n-1}+1}, \ldots, x_{2^n}).$$

Note that the function $f_{n-1}(x_1, \ldots, x_{n-1})$ can be transformed after a suitable substitution of the form

$$x_i = s_{2^k}(y_{1i}, \ldots, y_{2^k i})$$

to the form $s_{2^{n-1}}(z_1, \ldots, z_{2^{n-1}})$, where $z_i = y_{jk}$ with suitable j and k depending on i (see [3]).

Similarly we can transform g_n to the form

$$s_{2^{n-1}}(z_1, \ldots, z_{2^{n-1}})\cdot s_{2^{n-1}}(u_1, \ldots, u_{2^{n-1}}).$$

This operation is quasi-symmetrical and non-constant, because otherwise by putting $z_i = x$ and $u_i = y$ for $i = 1, 2, \ldots, 2^{n-1}$ we would obtain $x\cdot y = \text{const}$, a contradiction. It follows that this operation depends on all its variables and so does g_n. Finally observe that $f_n \neq g_n$.

We need also the following

LEMMA 5. *If $\mathfrak{A}$ is an algebra having at least two symmetrical essentially binary operations $+$ and $\cdot$, of which $+$ is idempotent, and $x\cdot x$ is a constant, then there are at least five distinct essentially ternary operations in $\mathfrak{A}$.*

Consider the operations $f_{11}, f_{12}, f_{13}, f_{31}, f_{32}, f_{33}$ and f_{41} as defined in the proof of Theorem II above. It is easy to see that the operations f_{1k} and f_{3k} are essentially ternary. To prove that the operation f_{41} is also essentially ternary consider the quasi-symmetrical operation $xy+zu$. It is obviously not constant, and it follows that f_{41} is essentially ternary. Moreover, observe that the operations f_{11}, f_{31} and f_{41} are all distinct. In fact, if f_{11} equals f_{31}, then putting $x = y$ we get $x+z = xz$. If $f_{11} = f_{41}$, then putting at first $z = x\cdot y$ and then $z = x+y$ we get also $x+y = x\cdot y$. If $f_{31} = f_{41}$, then identifying x and y we obtain $x\cdot z = c+z$ with a constant c, contrary to our assumptions.

If $+$ is not associative, then we obtain five different ternary operations: f_{1k} $(k = 1, 2, 3), f_{31}$ and f_{41}, because no operation f_{1k} is equal to one of the remaining four, as this would imply the symmetry of f_{1k} and the associativity of $+$. If the operation $+$ is associative, then we obtain also five different essentially ternary operations, namely f_{11}, f_{3k} ($k = 1, 2, 3$) and f_{41}. In fact, f_{31} cannot be symmetrical as this would imply

$$c = (x+y)(x+y) = \big(x+(x+y)\big)\cdot y = (x+y)y$$
$$= (y+x)y = (y+y)x = yx = xy,$$

contrary to our assumptions. It follows that none of the operations f_{3k} ($k = 2, 3$) can be equal to one of the remaining four, as this would clearly imply the symmetry of f_{31}. The lemma is thus proved.

Now observe that the equations $p(1, \boldsymbol{K}^{(2)}) = 2$ and $p(2, \boldsymbol{K}^{(2)}) = 4$ follow in the same way as the corresponding equations in theorem II. Using lemma 4 if $x\cdot x = x$, and using lemma 5 if $x \cdot x = \text{const}$, we obtain by (12) the inequality $p(n, \boldsymbol{K}^{(2)}) \geqslant 2(2^n-1)$. To end the proof, observe that the free algebra with n generators in the class described in (v) realizes equality in the last inequality. In fact, it has only the following operations: x, x^2, $x_1+\ldots+x_n$ and $x_1 \ldots x_n$ and the needed result follows from (12).

Now let $\boldsymbol{K}^{[2]}$ be the class of all algebras having at least two elements, and possessing at least two distinct symmetrical and idempotent essentially binary operations. We prove the following

THEOREM IV. *The following equations are true*:

$$p(1, \boldsymbol{K}^{[2]}) = 2, \quad p(2, \boldsymbol{K}^{[2]}) = 4, \quad p(3, \boldsymbol{K}^{[2]}) = 18.$$

(This theorem gives only a partial result for $p(n, \boldsymbol{K}^{[2]})$. We conjecture that the extremal algebra for this problem is the free distributive lattice with n generators. If it is so, then the calculation of $p(n, \boldsymbol{K}^{[2]})$ coincides with the unsolved problem of determining the cardinality of the free distributive lattice with n free generators, which goes back to Dedekind).

LEMMA 6. *If in an algebra $\mathfrak{A}$ (with $a(\mathfrak{A}) \geqslant 2$) there exist no other essentially j-ary operations for $j = 1, 2$ save $x, x+y, x\cdot y$, where $+$ and $\cdot$ are distinct idempotent, symmetrical, associative and essentially binary operations, then the equation*

$$x(x+y) = x+xy = x \tag{13}$$

holds true.

Proof. Note first that neither $x(x+y)$ nor $x+xy$ can be constant, as otherwise putting $x = y$ we would get $x = \text{constant}$. If $x(x+y) = y$, then

$$y = (x+y)(x+y+y) = x+y;$$

if $x(x+y) = xy$, then

$$x+y = (x+y)(x+y+x) = (x+y)x = xy;$$

in both cases a contradiction. Hence either $x(x+y) = x+y$ or $x(x+xy) = x$.

If $x+xy = y$, then

$$y = xy+xyy = xy,$$

and if $x+xy = x+y$, then

$$xy = xy+xyx = xy+x = x+y;$$

in both cases a contradiction. Hence either $x+xy = xy$ or $x+xy = x$.

If $x(x+y) = x+y$ and $x+xy = xy$, then

$$xy(x+y) = x(x+y) = x+y,$$

but on the other hand

$$xy(x+y) = xy+xy(x+y) = xy+x(x+y) = xy+x+y = xy+x = xy,$$

whence $xy = x+y$, a contradiction.

If $x(x+y) = x+y$ and $x+xy = x$, then

$$x = x+x(x+y) = x+x+y = x+y,$$

a contradiction. Finally, if $x(x+y) = x$ and $x+xy = xy$, then

$$x = x(x+xy) = xxy = xy,$$

again a contradiction. Thus it remains only the possibility $x(x+y) = x = x+xy$, as asserted.

The obtained result can be also stated in the following form:

(ix) *If an algebra $\mathfrak{A} = (X; +, \cdot)$ satisfies the assumptions of Lemma 6, then it is a lattice.*

LEMMA 7. *If $+$ and $\cdot$ are distinct symmetrical, idempotent and essentially binary operations in an algebra $\mathfrak{A}$ with $a(\mathfrak{A}) \geqslant 2$, and the operation $f_{31}(x, y, z) = (x+y)z$ is symmetrical, then there exists in $\mathfrak{A}$ an essentially binary operation which is different from $+$ and $\cdot$. An analogous statement is true if we replace the operation f_{31} by $f_{41}(x, y, z) = (x \cdot y)+z$.*

Proof. If f_{31} is symmetrical, then

$$x+y = (x+y)(x+y) = \big(x+(x+y)\big)y.$$

Let $k(x, y) = x+(x+y)$. The operation k is not constant because it is idempotent, and the algebra contains at least two elements. If $k(x, y) = x$ or $k(x, y) = y$, then the above equation implies $x+y = xy$ or $x+y = y$. It follows that k is essentially binary. If $k(x, y) = xy$ or $x+y$, then

$$x+y = \big(x+(x+y)\big)y = \big(y+(x+y)\big)y = (y+y)(x+y)$$
$$= (x+y)y = (y+y)x = xy,$$

a contradiction.

The replacement of f_{31} by f_{41} does not affect the result as $x+y$ and $x\cdot y$ can be interchanged.

LEMMA 8. *Under the assumptions of Lemma 6 there exist in $\mathfrak{A}$ at least nine different essentially ternary operations.*

Proof. Lemma 4 shows that the operations f_{11}, f_{21}, f_{3k} and f_{4k} $(k = 1, 2, 3)$ are different and essentially ternary. We show that the operation

$$f_5(x, y, z) = (x+y)(x+z)(y+z)$$

is essentially ternary and distinct from f_{ik} $(3 \leqslant i \leqslant 4, 1 \leqslant k \leqslant 3)$ and f_{11}, f_{21}. In fact, it is clearly symmetrical and not constant, hence it is essentially ternary. It is not equal to f_{3k} or f_{4k} as they are not symmetrical in view of Lemma 3 (observe that if f_{3k} is symmetrical, then f_{31} is too, and the same applies to f_{4k}), and is not equal to f_{11} or f_{21} as in these cases putting $x = y$ we would obtain a contradiction with Lemma 6. Lemma 8 is thus proved.

The proof of equations $p(1, \boldsymbol{K}^{[2]}) = 2$ and $p(2, \boldsymbol{K}^{[2]}) = 4$ proceeds in the same way as the proof of the corresponding equations in Theorem II. Now observe that for any algebra $\mathfrak{A}$ in $\boldsymbol{K}^{[2]}$ we have $\omega_1 \geqslant 1$, $\omega_2 \geqslant 2$, and $\omega_3 \geqslant 8$. (This last inequality follows from Lemma 4). Lemma 7 implies that if one of the operations f_{31}, f_{41} is symmetrical, then $\omega_2 \geqslant 3$ and (12) implies $a(\mathfrak{A}) \geqslant 20$. If both operations f_{31}, f_{41} are not symmetrical and either $+$ or $\cdot$ is not associative, then, by Lemma 2, we have $\omega_3 \geqslant 10$, and (12) implies $a(\mathfrak{A}) \geqslant 19$. If both operations $+$ and $\cdot$ are associative, and there exists some non-constant and non-trivial unary or binary operation distinct from $+$ and $\cdot$, then, similarly, we obtain $a(\mathfrak{A}) \geqslant 20$. Finally, if the assumptions of Lemma 6 are satisfied, then Lemma 8 implies $\omega_3 \geqslant 9$, hence, by (12), $a(\mathfrak{A}) \geqslant 18$. It results that $p(n, \boldsymbol{K}^{[2]})$ is at least equal to 18. To end the proof observe that the free distributive lattice with three generators is in $\boldsymbol{K}^{[2]}$ and has 18 elements. The theorem is thus proved.

3. If $\boldsymbol{K}$ is a class of algebras, then we define the function $q(n, \boldsymbol{K})$ by the formula

$$q(n, \boldsymbol{K}) = \max\{\iota(\mathfrak{A}) : \mathfrak{A} \in \boldsymbol{K},\ a(\mathfrak{A}) = n\}.$$

We prove

THEOREM V. *For $n \geqslant 2$ we have $q(n, \boldsymbol{K}_c^2) = [(n-1)^{1/2}]$.*

Proof. Theorem I implies the inequality $q(n, \boldsymbol{K}_c^2) \leqslant [(n-1)^{1/2}]$. To prove that equality occurs here take for $n = 2, 3, 4$ the linearly ordered lattice of n elements with usual operations to which an algebraic constant, the greatest element, has been added. For n exceeding 4 proceed as follows: let $m = [(n-1)^{1/2}]$ and consider the algebra $\mathfrak{C}^{(m)}$, as defined below (ii).

To the carrier of this algebra add a new set E having $n-(m^2+1)$ elements and define in the sum-set operations $x+y$ and $x\cdot y$ in the same way as it was done for $\mathfrak{C}^{(m)}$ (see § 1). It is easy to see that the algebra obtained in this way proves the wanted equation.

THEOREM VI. *For* $n \geqslant 13$ *we have*

$$q(n, \boldsymbol{K}_w^2) = \max\left(m: 2^m - 1 + \binom{m}{2} + m \leqslant n\right),$$

whereas for $n = 4, 5, \ldots, 12$ *we have*

$$q(n, \boldsymbol{K}_w^2) = 2 \quad \text{and} \quad q(2, \boldsymbol{K}_w^2) = q(3, \boldsymbol{K}_w^2) = 1.$$

Proof. Theorem II implies that the left-hand sides of the asserted equations do not exceed the right-hand sides. To prove that the equations are true, take for $n = 2, 3$ the linearly ordered lattices with n elements, and for $n = 4, 3, \ldots, 12$ take the lattices of the form $\{a_1, \ldots, a_n; \cup, \cap\}$, where $a_1 \cap a_2 = a_3$, $a_1 \cup a_2 = a_4$ and the elements $a_4, a_5, \ldots, a_n$ are linearly ordered according to their indices. For the case $n \geqslant 13$ consider the algebra $\mathfrak{A}_m$ as defined in (iv), with

$$m = \max\left\{k: 2^k - 1 + \binom{2}{k} + k \leqslant n\right\},$$

and add to S a linearly ordered set P of

$$r = n - 2^m + 1 - \binom{m}{2} - m$$

elements. Assume, moreover, that $x < p$ holds for $p \in P$ and every x from S. The obtained algebra with the operation $x\cdot y$ defined as in (iii) and (iv) furnishes an example proving the asserted equation.

THEOREM VII. *For* $n \geqslant 14$ *we have*

$$q(n, \boldsymbol{K}^{(2)}) = \max\{m: 2(2^m - 1) \leqslant n\},$$

whereas $q(n, \boldsymbol{K}^{(2)}) = 1$ *for* $n = 2, 3$ *and* $q(n, \boldsymbol{K}^{(2)}) = 2$ *for* $n = 4, 5, \ldots, 13$.

Proof. In view of Theorem III only the extremal algebras have to be shown. For n not exceeding 13, the construction is the same as in the foregoing theorem. For $n \geqslant 14$, take the free algebra $\mathfrak{F}$ with

$$m = \max\{p: 2(2^p - 1) \leqslant n\}$$

free generators in the class described in (v) and add to its carrier a set P of $n - 2(2^m - 1)$ elements. Order this set linearly and assume that $P > x$ holds for $p \in P$ and every x belonging to the carrier of $\mathfrak{F}$. Moreover, let $r(x)$ occurring in (v) be prolonged on P by means of $r(p) = p$ $(p \in P)$.

If in this enlarged algebra we define $x \cdot y = r(x) + r(y)$, then we obtain the wanted example.

Observe that one can easily extend Theorems I, II, III and IV to the corresponding classes of algebras with s different binary symmetrical operations.

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INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES

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