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ON A NEW FINITE-DIFFERENCE SCHEME FOR THE NON-STATIONARY NAVIER-STOKES EQUATIONS

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In this paper a new implicit finite-difference scheme for a general initial value problem for the Navier-Stokes equations is given. The paper is a continuation of paper [3], where two different implicit schemes have been presented for the same problem. However, the numerical realization of schemes given in [3] would be complicated: these are 7-point schemes with four unknown functions. The aim of this paper is to show that a partial simplification may be gained by using a scheme modelled on those appearing in the so called IAD (implicit alternating direction) - method (see, e.g., [1]). By introducing two auxiliary "intermediate" velocities (denoted by v and w in equations (5)-(8) below) the original Navier-Stokes equations may be replaced by a system of $3\times 3+1$ difference equations and in a part of them a complete separation of variables is attained: they present 3-point one-dimensional schemes. The remaining equations present a 4-point scheme.

Due to the character of equations under consideration we do not hope to preserve much of the merits of IAD-method. It is shown that the proposed scheme converges to a weak solution of the problem under consideration under the same assumption concerning the mesh sizes of the lattice as for the explicit scheme for the heat equation. The idea of the proof is the same as in [3]: it is based on an a priori estimate (see formula (21)) which is a modified classical energy inequality for solutions of Navier-Stokes equations. This method restricts the form of admissible difference schemes, in particular, the term which corresponds to non-linear term in original Navier-Stokes equations should be chosen so as to bring no contribution into the basic a priori estimate. Therefore, e.g., the explicit schemes should be excluded from our considerations.

1. Let Ω be a bounded domain in E^3 with the boundary S. Consider for $(x, t)\in Q = \Omega\times[0, T]$, x denoting a point of E^3, $x = (x^1, x^2, x^3)$, and T — a positive constant, the following initial value problem for an unknown vector-valued function $\mathscr{U} = \mathscr{U}(x, t)$ and a scalar function $\mathscr{P} = \mathscr{P}(x, t)$:

$$(1) \qquad \frac{\partial \mathscr{U}}{\partial t} - \nu \Delta \mathscr{U} + (\mathscr{U} \cdot \nabla)\mathscr{U} = -\nabla \mathscr{P} + \mathscr{F}, \qquad \nabla \cdot \mathscr{U} = 0,$$

$$\mathscr{U}|_s = 0, \qquad \mathscr{U}|_{t=0} = \mathscr{A}.$$

∇ denotes the gradient, $\nabla \cdot \mathscr{U} \equiv \sum_i (\partial \mathscr{U}^i / \partial x^i)$ — the divergence of $\mathscr{U}$, $\mathscr{U}^i$ $(i = 1, 2, 3)$ — the i-th component of $\mathscr{U}$, $\mathscr{F} = \mathscr{F}(x, t)$ and $\mathscr{A} = \mathscr{A}(x)$ are given vector-valued functions (we shall say simply *vectors*) defined on Q and Ω respectively. ν is a positive constant.

Let $\mathscr{M}$ denote the class of $C^\infty(Q)$ vectors Φ which, for each $t \in [0, T]$, are divergence free, i.e. $\nabla \cdot \Phi = 0$, and which have compact supports in $\Omega \times [0, T)$. Let further $\mathring{J}(\Omega)$ be the closure under $L_2(\Omega)$-norm of the class of $C^\infty(\Omega)$ vectors which are divergence free and have compact supports in Ω. Assume $\mathscr{F} \in L_2(Q)$, $\mathscr{A} \in \mathring{J}(\Omega)$. A vector $\mathscr{U}$ will be called a *weak solution* (in the sense of E. Hopf) of the problem (1) if $\mathscr{U}$ is, for each $t \in [0, T)$, an element of $\mathring{J}(\Omega)$, if $\nabla \mathscr{U} \in L_2(Q)$, and if for all $\Phi \in \mathscr{M}$ the following identity holds true:

$$(2) \qquad \int_0^T dt \int_\Omega \Big\{ \mathscr{U} \frac{\partial \Phi}{\partial t} - \nu \sum_i \frac{\partial \mathscr{U}}{\partial x^i} \frac{\partial \Phi}{\partial x^i} - \sum_i \mathscr{U}^i \frac{\partial \mathscr{U}}{\partial x^i} \Phi + \mathscr{F} \Phi \Big\} dx +$$

$$+ \int_\Omega \mathscr{A}(x) \Phi(x, 0)\, dx = 0.$$

In the last formula $\mathscr{U}(\partial \Phi / \partial t)$ denotes the ordinary scalar product of vectors $\mathscr{U}$ and $\partial \Phi / \partial t$ and similarly in the remaining terms.

2. Consider in $E^3 \times [0, T]$ a rectangular grid $G_{h,k}$ of points with coordinates $x^i \equiv \pm nh$, $t = nk$, where h and k are positive constants, $n = 0, 1, 2, \ldots$ Let K be an elementary cube of the grid of the x-space,

$$K = \Big\{x\colon\ x = x' + \sum_i \lambda_i e^i,\ 0 \leqslant \lambda_i \leqslant 1\Big\},$$

where x' is a point of $G_{h,k}$ and e^i — unit vector parallel to the x^i-axis. Let $\bar{K}$ denote the closure of K, Ω^n—a closed domain of the plane $t = nk$ equal to the union of all those $\bar{K}$ which lie within Q, and let finally S^n denote the boundary of Ω^n. We shall denote the sets $\Omega^n \cap G_{h,k}$, $S^n \cap G_{h,k}$ and $(\Omega^0 \times [0, T]) \cap G_{h,k}$ by ω^n, s^n and q respectively and we shall call s^n the *boundary* of ω^n. The sets ω^0 and s^0, when considered as subsets of Ω, will be denoted simply by ω and s, respectively.

All functions considered on the subsets of $G_{h,k}$ will be denoted by small letters, all functions defined on Q — by capital letters.

Let a function v be given. We shall use in the sequel the following notation:

$$\overset{\pm i}{v}(x,t) = v(x \pm h e^i, t),$$

$$v_i(x,t) = h^{-1}[\overset{+i}{v}(x,t) - v(x,t)],$$

$$v_{\bar{i}}(x,t) = \overset{-i}{v_i}(x,t),$$

$$\bar{v}(x,t) = v(x, t-k).$$

Here e^i, as before, is unit vector parallel to the x^i-axis. We adopt also the standard vector and L_2-space notation and we shall write, for any vectors u and v:

$$uv = \sum_i u^i v^i, \qquad u_x v_x = \sum_i u_i v_i,$$

$$(u, v)_n = h^3 \sum uv, \qquad \|u(n)\|^2 = (u, u)_n,$$

$$(u_x, v_x)_n = h^3 \sum u_x v_x, \qquad \|u_x(n)\|^2 = h^3 \sum u_x v_x,$$

the last three sums $\sum$ being taken (here as always in the sequel in similar expressions) over all those points of ω^n, where the expression under $\sum$-sign is defined, i. e., where all the points involved in this expression lie within ω^n.

Let K be an elementary cube introduced before, Σ—its boundary, and let K_λ (resp. Σ_λ), $0 \leqslant \lambda \leqslant h$, denote a cube (resp. surface) obtained as a result of parallel shifting of K (resp. Σ) along the vector $e^1 + e^2 + e^3$ at distance $\lambda\sqrt{3}$. Let $\mathscr{W}$ be any divergence free vector defined in Ω which vanishes near S. Extend $\mathscr{W}$ to all E^3 by putting $\mathscr{W} = 0$ outside Ω, and denote this extension with the same letter $\mathscr{W}$. We define on $G_{h,k}$ a difference approximation w of $\mathscr{W}$ by putting in the corner x' of K (compare definition of K)

$$w^i(x') = h^{-3} \int_0^h d\lambda \int_{\Sigma_\lambda^i} \mathscr{W}^i d\Sigma, \tag{4}$$

where Σ_λ^i denotes that face of K which lies in the plane $x_i = x'_i + \lambda$. To assure vanishing of w along the boundary s we make the following

Assumption (I). h is so small that for any point $x' \in s$ the set $\bigcup_\lambda K_\lambda$, $\lambda \in [0, h]$, lies outside of the support of $\mathscr{W}$.

The vector w thus defined satisfies the equation

$$\operatorname{div} w \equiv \sum_i w_i^i = 0$$

and this follows from the identity

$$\operatorname{div} w = h^{-4} \int_0^h d\lambda \int_{\Sigma_\lambda} \mathscr{W}_n d\Sigma,$$

where $\mathscr{W}_n$ denotes the outward normal component of the vector $\mathscr{W}$ along Σ_λ. The inner integral of the last formula vanishes due to the assumption $\nabla \cdot \mathscr{W} = 0$.

Few other approximations, besides that one described above, will be needed in the sequel (similarly as in [3]).

Given a function u (or $v, w, \varphi, \ldots$), we shall denote by U (or $V, W, \Phi, \ldots$) a function which is constant within each parallelepiped $K_n = K \times [nk, (n+1)k)$, K having the previous meaning, and equal there to the value of u (or $v, w, \varphi, \ldots$) in the corner (x', nk) of K_n if $\bar{K}_n \subset Q$. Otherwise we put $U\ (= V = W = \ldots) = 0$. In case of a function u dependent only on x, the function U, defined above, will be considered, if needed, as function defined only in x-space.

Another approximation $\mathfrak{U}$ (respectively $\mathfrak{V}, \mathfrak{W}, \ldots$) of a function u (respectively $v, w, \ldots$) will be defined as follows: $\mathfrak{U}$ is linear with respect to each x^i separately whereas $\mathfrak{U}$ is constant with respect to t within each elementary parallelepiped K_n and coincides with u in all points of the grid $G_{h,k}$ if $\bar{K}_n \subset Q$. Otherwise we put $\mathfrak{U} = 0$ within that K_n. Both extensions U and $\mathfrak{U}$ have been introduced in [4] and the proofs of all their properties needed below may be found there.

Finally, the vector $\mathscr{F}(x, t)$ appearing in (1) will be approximated by the vector f defined as follows: the value of f in the grid point (x', nk) is equal to

$$h^{-3} k^{-1} \int_{K_n} \mathscr{F} dx\, dt.$$

If we extend $\mathscr{F}$ to all spaces of (x, t) putting $\mathscr{F} = 0$ outside of Q, then the last formula defines f on all $G_{h,k}$. Note that the approximation $\mathscr{F}$ of f is strongly convergent to $\mathscr{F}$ in $L_2(Q)$ if only $\mathscr{F} \in L_2(Q)$. Similarly, the approximation W of w, w being defined by formula (4), converges strongly in $L_2(\Omega)$ to $\mathscr{W}$, if only $\mathscr{W} \in L_2(\Omega)$. The last assertion follows from the identity

$$\mathscr{W}^i(x) - W^i(x) = h^{-3} \int_{R_i} [\mathscr{W}^i(x) - \mathscr{W}^i(y)] dy$$

valid for any x. R_i in the last formula denotes the domain swept out by Σ_λ^i, when λ ranges over the interval $(0, h)$.

3. Let us take a sequence $\mathscr{A}^{(m)}$ of regular divergence free vectors with compact supports in Ω converging strongly in $L_2(\Omega)$ to $\mathscr{A}$. Then, for each m, form a vector $a^{(m)}$ in accordance with (4) and its approximation $A^{(m)}$ by choosing $h = h_m$ so that: (a) $h_m \to 0$ for $m \to \infty$, (b) for each m condition (I) is satisfied with $h = h_m$ and $\mathscr{W} = \mathscr{A}^{(m)}$, (c) $\|A^{(m)} - \mathscr{A}\|_{L_2(\Omega)} \to 0$ for $m \to \infty$. A sequence $\{h\}$ will be said *to satisfy assumption* (II) if its elements $h = h_m$ satisfy (a)-(c).

Denote by u and p the difference analogues of the original $\mathscr{U}$ and $\mathscr{P}$ of (1) and introduce two auxiliary vectors v and w. Consider, for any $t = nk$, $1 \leqslant n \leqslant N$, $N = [T/k]$, the following system of finite difference equations

$$(5) \qquad k^{-1}(v-\bar{u}) - \nu(v_{1\bar{1}} + \bar{u}_{2\bar{2}} + \bar{u}_{3\bar{3}}) + \tfrac{1}{2}\{\bar{u}_1^1 \overset{+1}{v} + \bar{u}^1(v_1 + v_{\bar{1}})\} = 0,$$

$$(5_1) \qquad v|_{s^n} = 0,$$

$$(6) \qquad k^{-1}(w-v) - \nu(w_{2\bar{2}} - v_{2\bar{2}}) + \tfrac{1}{2}\{v_2^2 \overset{+2}{w} + v^2(w_2 + w_{\bar{2}})\} = 0,$$

$$(6_1) \qquad w|_{s^n} = 0,$$

$$(7) \qquad k^{-1}(u-w) - \nu(u_{3\bar{3}} - w_{3\bar{3}}) + \tfrac{1}{2}\{w_3^3 \overset{+3}{u} + w^3(u_3 + u_{\bar{3}})\} = -\operatorname{grad} p + f,$$

$$(7_1) \qquad u|_{s^n} = 0,$$

$$(8) \qquad \operatorname{div} u = 0,$$

where $\operatorname{grad} p$ denotes the vector with components $p_{\bar{i}}$. Moreover, u is assumed to satisfy the following initial condition:

$$(9) \qquad \bar{u}|_{n=1} = a^{(m)}.$$

The proposed system is such that, assuming u and p to be known on ω^{n-1}, we find u and p on ω^n by successive solving subsystems (5), (5_1), then (6), (6_1) and finally (7)-(8) plus an additional equation (see below, (10)). The vectors v and w play here only an auxiliary role. Due to them the desired simplification has been gained.

Equations (5), (6) and (7) are taken, for any n, $1 \leqslant n \leqslant N$, in all interior points of ω^n and equations (8) in all points of the set ω^* which consists of all interior points of ω^n plus that part s^* of s^n where (8) is defined, i.e., where the four points entangled in (8) lie within ω^n, and at least one of those points lies outside of s^n. The reason for the last restriction is that if all four points involved in (8) lie on s^n, then (8) is a linear combination of equations of (7_1).

System (5)-(8) thus defined consists of so many equations as is the number of values of v, w, u, p appearing there. This is clear as far as either the unknowns v, w, u and equations (5)-(7_1) or the unknown p and equations (8) taken in interior points of ω^n are considered. The remaining equations (8) are taken in all points of set s^*. Now in equations (7) there appear boundary values of p and these are taken, as is easily seen, in and only in points of the same s^*. This together with the preceding remark proves our assertion. However, equations (8) are linearly dependent due to the relation

$$\sum_{\omega^*} \operatorname{div} u = h^{-1} \sum_{s^n} \sum_{i} \varepsilon u^i = 0, \qquad \varepsilon = \pm 1 \text{ or } 0,$$

valid for any $n \leqslant N$. Therefore we attach to our system an additional equation

$$\sum_{\omega^*} p = 0. \tag{10}$$

In some cases single equation (10) may not suffice. It does, however, when ω is connected, which means the following: each pair of points of $\omega \setminus s$ may be joined by an "h-chain" (with h steps) consisting of points of $\omega \setminus s$. To exclude the situation when ω or, which is the same, ω^n is not connected, we replace ω by a new set ω': it results by attaching to ω some additional points of the grid lying along all those "narrows" of the domain Ω which have caused the decomposition of ω into disjoint components. In the case when such a completion of ω to ω' is needed, the system (5)-(10) is considered on this new set ω' with v, w, u satisfying homogeneous boundary conditions along the boundary of ω'. Then we additionally assume Ω and ω' to be so that mes $(\Omega' \setminus \Omega) \to 0$, with $h \to 0$, where Ω' denotes the union of all elementary cubes with at least one corner in ω'.

4. Let U be the approximation of a solution u of system (5)-(8) described in section 2.

THEOREM 1. *Let $\{h\}$ be any sequence of positive numbers tending to zero, subject to assumption* (II) *formulated at the beginning of section* 3. *Put $k = ah^2$ with $a \leqslant A$, where A will be defined later* (*see formula* (17)). *Then*

(i) *system* (5)-(10) *is uniquely solvable for any pair $h, k, k = ah^2$, $h \in \{h\}$,*

(ii) *from the set $\{U\}$ of U's corresponding to $h \in \{h\}$ a subsequence may be chosen which converges strongly in $L_2(Q)$ to a weak solution of problem* (1).

5. The proof of Theorem 1 will be based on some a priori estimate of solutions of system (5)-(10). In this section we shall derive this estimate.

We begin with some auxiliary identities. We have, first of all, the identity

$$h^3 \sum u_i v = -h^3 \sum u v_{\bar{i}} \tag{11}$$

valid for any pair of functions u, v defined on ω^n if only uv vanishes on s^n. In particular, we have

$$h^3 \sum u_i = 0, \quad h^3 \sum \sum_i u v_{i\bar{i}} = -(u_x, v_x)_n \tag{12}$$

for any pair u, v, if only u vanishes on s^n.

Due to the identity

$$(uv)_i = u_i \overset{+1}{v} + uv_i \tag{13}$$

we can write:

$$(u \overset{-i}{v} v)_i = u_i v \overset{+i}{v} + u(\overset{-i}{v} v)_i = [u_i \overset{+i}{v} + u(v_i + v_{\bar{i}})]v.$$

Therefore, by (12), we have

$$h^3 \sum [u_i \overset{+i}{v} + u(v_i + v_{\bar{i}})]v = 0,$$

if only $u \overset{-i}{v} v|_{s^n} = 0$. We have also, due to (11),

$$h^3 \sum u \operatorname{grad} p = 0$$

for any u satisfying the equation $\operatorname{div} u = 0$ in ω^n and vanishing on s^n. Note also the following identity

$$2h^3 \sum (u - v) u = \|u(n)\|^2 - \|v(n)\|^2 + \|u(n) - v(n)\|^2.$$

6. Form the inner products (in the sense of formula (3)) of equations (5), (6) and (7), taken at any fixed $t = nk$, $1 \leqslant n \leqslant N$, with $2kv$, $2kw$, $2ku$, resp., taken at the same t. Making use of some of the identities given in the preceding section, we get the following three equations

$$\left\{\begin{aligned} \|v\|^2 - \|\bar{u}\|^2 + \|v - \bar{u}\|^2 + 2\nu k\{\|v_1\|^2 + (v_2, \bar{u}_2)_n + (v_3, \bar{u}_3)_n\} &= 0, \\ \|w\|^2 - \|v\|^2 + \|w - v\|^2 + 2\nu k\{\|w_2\|^2 - (w_2, v_2)_n\} &= 0, \\ \|u\|^2 - \|w\|^2 + \|u - w\|^2 + 2\nu k\{\|u_3\|^2 - (u_3, w_3)_n\} &= 2k(f, u)_n, \end{aligned}\right. \tag{14}$$

where all u, v, w, if not explicitely written down, are taken at the same $t = nk$.

If we now sum up, side by side, equations (14) and then perform some simple transformations, we get the identity

$$\|u(n)\|^2 - \|u(n-1)\|^2 + S^2(n) + 2\nu k H(n) = 2k(f, u)_n, \tag{15}$$

where

$$S^2(n) = \|v(n)-u(n-1)\|^2+\|w(n)-v(n)\|^2+\|u(n)-w(n)\|^2,$$
$$H(n) = \|v_1(n)\|^2+\|w_2(n)\|^2+\|u_3(n)\|^2+(\bar{u}_2-w_2, v_2)_n+$$
$$+(\bar{u}_3-w_3, v_3)_n+(v_3-u_3, w_3)_n.$$

If we now make use of the identity

$$\|u_i\|^2 = \|v_i\|^2+(u_i+v_i, u_i-v_i),$$

which is valid for any functions u, v, we can rewrite $H(n)$ in the form

$$H(n) = \tfrac{1}{3}\{D^2(n)+\sum(\tilde{u}_i, \tilde{v}_i-\tilde{w}_i)_n\}. \tag{16}$$

Here

$$D^2(n) = \|v_x(n)\|^2+\|w_x(n)\|^2+\|u_x(n)\|^2,$$

$\tilde{u}$ denotes any one of the vectors v, w, u, and $\tilde{v}$, $\tilde{w}$ — any two of the vectors $\bar{u}$, v, w, u, all taken at $t = nk$, the sum (16) being extended over $\tilde{u}$, $\tilde{v}$, $\tilde{w}$ and i. It contains a finite number of terms, some of them may be repeated, yet the form of $H(n)$ is the same for all n. Applying now the inequality

$$|(u_i, v_i-w_i)| \leqslant \beta^{-1}\|u_i\|^2+\beta h^{-2}\|v-w\|^2,$$

valid for any positive β and any functions u, v, w, we can estimate the second term in (16) by $\frac{1}{6}D^2(n)+Mh^{-2}S^2(n)$ with some positive constant M, which is independent of $n, h, \bar{u}, v, w$ and u. If we assume

$$kh^{-2} \leqslant A, \qquad A = (4M\nu)^{-1}, \tag{17}$$

we then get from (15) the inequality

$$\|u(n)\|^2-\|u(n-1)\|^2+\tfrac{1}{2}S^2(n)+\nu k/3\,D^2(n) \leqslant 2k(f, u)_n. \tag{18}$$

Applying Cauchy-Schwartz' inequality to the right-hand member of the last relation we get

$$\|u(n)\|^2-\|u(n-1)\|^2 \leqslant 2k\|f(n)\|\,\|u(n)\|.$$

Hence $\|u(n)\| \leqslant \|u(n-1)\|+2k\|f(n)\|$ and, consequently,

$$\|u(n)\| \leqslant \|u(0)\|+2k\sum_{1\leqslant m\leqslant n}\|f(m)\|.$$

We apply the last inequality to the right-hand member of (18) and then we sum up the inequalities obtained in this way. As a result we get the inequality

$$\|u(n)\|^2+\sum_{1\leqslant l\leqslant n}\{\tfrac{1}{2}S^2(l)+\nu k/3\,D^2(l)\} \leqslant C_0 \tag{19}$$

valid for any $n \leqslant N$ if only $kh^{-2} \leqslant A$. The constant C_0, as is easily seen, may be estimated as follows:

$$C_0 \leqslant 5\Big(k \sum_{1\leqslant l\leqslant N} \|f(l)\|\Big)^2 + \|u(0)\|^2. \tag{20}$$

Therefore C_0 is less than a constant depending only on $\|\mathscr{F}\|_{L_2(Q)}$ and $\|\mathscr{A}\|_{L_2(\Omega)}$.

From (19) and from $\|v(n)\|^2 + \|w(n)\|^2 \leqslant 4\|u(n)\|^2 + 6S^2(n)$, the last relation being an immediate consequence of the triangle inequality, we get finally the needed basic inequality

$$\max_{n\leqslant N}\{\|v(n)\|^2 + \|w(n)\|^2 + \|u(n)\|^2\} + 6\sum_{1\leqslant l\leqslant N} S^2(l) + 8\nu k \sum_{1\leqslant l\leqslant N} D^2(l) \leqslant C \tag{21}$$

with $C = 24C_0$, valid for the solutions of system (5)-(10) if only the condition (17) is satisfied.

7. Part (i) of Theorem 1 is a consequence of the fact that the homogeneous system of equations corresponding to (5)-(10) has only a trivial solution and this immediately follows from identities (14) if we put there $\bar{u}|_{n=1} = f = 0$. Note that identities (14) were derived without assuming (17).

8. Let u be a solution of system (5)-(10) corresponding to h and k satisfying condition (17), and let $\mathfrak{U}$ denote its extension defined in section 2. It is easily seen that (21) implies the inequality

$$\sup_{0\leqslant t\leqslant Nk} \|\mathfrak{U}(t)\|^2 + \int_0^{Nk} \|\nabla\mathfrak{U}(t)\|^2 dt \leqslant C \tag{22}$$

with some constant C (the same letter C will denote, when used in different places, different constants). In the last formula $\|\ \|$ denotes the ordinary $L_2(\Omega)$-norm.

In view of (22) the sets $\{\mathfrak{U}\}$ and $\{\nabla\mathfrak{U}\}$ are weakly compact in $L_2(Q)$, therefore we can choose a subsequence $\{h'\}$ from $\{h\}$ such that the corresponding sequences $\{\mathfrak{U}'\}$ and $\{\nabla\mathfrak{U}'\}$ are weakly convergent to some $\mathscr{U}$ and to $\nabla\mathscr{U}$, respectively, when $h' \to 0$. Consider now the approximations U' (see section 2) of u' ($'$ denotes that only terms of the sequence $\{h'\}$ are considered). Due to inequality (21) we can assert that the sequences $\{U'\}$ and $\{U'_i\}$ are also weakly convergent in $L_2(Q)$ to the same limits $\mathscr{U}$ and $\partial\mathscr{U}/\partial x^i$, respectively (see [4], Theorem 20, p. 52), when $h' \to 0$.

9. We shall show still more: $\{h'\}$ may be chosen so that $\{U'\}$ converge strongly in $L_2(Q)$ to $\mathscr{U}$ when $h' \to 0$. This will be done by adapting Hopf's procedure (see [2] or [5]) to the case under consideration.

Denote by $\mathcal{N}$ the class of all regular and divergence free vectors with compact supports in Ω. For any fixed $\theta^* \in \mathcal{N}$ form ϑ in accordance with (4) assuming h to be so small as to assure the vanishing of ϑ on s. Now sum up, side by side, equations (5), (6) and (7), all taken at the same $t = nk$, and then form the inner product (in the sense of (3)) of the result with $k\vartheta$. Transforming the terms containing second order differences by using identity (12), we can write the result in the form

$$(u-\bar{u}, \vartheta)_n = kR(n), \tag{23}$$

where $R(n)$ is the sum of terms of the following forms: $(\tilde{u}_i, \vartheta^j_i)_n$, $\frac{1}{2}(\tilde{u}\tilde{w}_i, \vartheta^j)_n$ and $(f, \vartheta)_n$. Here $\tilde{u}$ denotes any component of $\bar{u}$, v, w, u, whereas $\tilde{w}$ denotes either any of these four functions or the result of their shifting at distance h parallelly to any x^i-axis. The form of $R(n)$ is independent of n. It is easily seen that we have the estimation

$$|R(n)| \leqslant C_\vartheta \Big\{D(n)\Big(1+\sum_{\tilde{u}} \|\tilde{u}(n)\|\Big) + \|f(n)\|\Big\} \tag{24}$$

with a constant C_ϑ depending only on $\|\vartheta_x\|$ and $\max\limits_{x\in\omega}|\vartheta(x)|$ ($|\vartheta|$ denoting Euclidean length of the vector ϑ) that is on $\|\nabla\theta^*\|_{L_2(\Omega)}$ and $\sup\limits_{x\in\Omega}|\theta^*(x)|$. Due to (24) and (21) we have

$$k\Big|\sum_{n'\leqslant n\leqslant n''} R(n)\Big| \leqslant C[(n''-n')k]^{1/2}$$

with C depending only upon C_ϑ and the constant of inequality (21), and this combined with (23) leads to the inequality

$$|\psi(t'')-\psi(t')| \leqslant C\sqrt{t''-t'}, \tag{25}$$

where $t' = n'k$, $t'' = n''k$, and

$$\psi(t) = (u, \vartheta)_n = \int\limits_\Omega U(x, t)\,\theta(x)\,dx, \qquad nk \leqslant t < (n+1)k, \tag{26}$$

θ in the last formula denoting the approximation of ϑ (see section 2).

Let ψ^c denote a continuous function of t linear on each interval $I_n = [nk, (n+1)k]$ and coinciding with ψ on both ends of each I_n. The functions ψ^c are, for fixed ϑ and $h \in \{h\}$, equicontinuous and uniformly bounded as it is seen from (25) and from the inequality $|\psi(nk)| \leqslant \|u(n)\|\|\vartheta\| \leqslant C$. Making use of Arzelà's theorem we can extract from $\{h\}$ a subsequence $\{h^*\}$ such that the functions ψ^c (as well as ψ corresponding to $h \in \{h^*\}$ are uniformly convergent on $[0, T)$ when $h \to 0$.

What we have now to do is just to repeat Hopf's reasonings (see [2]). Let $\theta^{(m)}$, $m = 1, 2, \ldots$, be a set of vectors of class $\mathcal{N} \cap C^1(\Omega)$, strongly dense in $\mathring{J}(\Omega)$. For each m, we construct the approximation

$\vartheta^{(m)}$ of $\theta^{(m)}$ in accordance with (4) and then $\psi^{(m)}$ in accordance with (26) assuming $h \in \{h^*\}$ to be sufficiently small, say $h \leqslant h(m)$, in order to satisfy the condition $\vartheta^{(m)}|_s = 0$. Let $\{h^{(m)}\}$ be, for any fixed m, a subsequence of $\{h'\}$, with all $h^{(m)}$ not greater than $h(m)$, such that the functions $\psi^{(m)}$ are uniformly convergent on $[0, T)$ when $h^{(m)} \to 0$. Assume, moreover, that for each m we have $\{h^{(m)}\} \subset \{h^{(m-1)}\}$. The application of standard diagonal process provides us with a subsequence of $\{h\}$ (it will be denoted equally by $\{h\}$) such that the sequence of corresponding functions $\psi^{(m)}$ is convergent for each m and each t of the interval $[0, T)$ when $h \to 0$. Consider the approximations U of solutions u corresponding to $h \in \{h\}$. The last result implies weak convergence $U \to \mathscr{U}$ for each $t \in [0, T)$: we have only to note that $\mathscr{U}$ lies in the closure of $\mathscr{N}$, i.e. in $\overset{\circ}{J}(\Omega)$, for each $t \in [0, T)$, and this follows from the easily derivable orthogonality of $\mathscr{U}$ to gradients of all functions which are regular in Ω and vanish near S. On the other hand, due to results of [4] (loc. cit.), it follows from the weak convergence $U \to \mathscr{U}$ and from (21) that also corresponding approximations $\mathfrak{U}$ converge weakly to $\mathscr{U}$ for each $t \in [0, T)$. Hence and from (22), applying Hopf's lemma (see [2], p. 172), we deduce the strong convergence of $\mathfrak{U}$ and therefore also of U (applying again [4], loc. cit.) to $\mathscr{U}$.

10. It has remained only to show that $\mathscr{U} = \lim U$ is a weak solution of the problem (1), i.e. that $\mathscr{U}$ satisfies the identity (2) for any $\Phi \in \mathscr{M}$. Take any $\Phi \in \mathscr{M}$ and construct its approximation $\varphi = \varphi(x, t)$, for any t, by using formula (4). We assume additionally h to be so small that $\varphi|_{S^n} = 0$ for any $n \leqslant N$ and that $\varphi \equiv 0$ for $t \geqslant (N-1)k$. We proceed as before when deriving (23) with $k\vartheta(x)$ replaced now by $\varphi\big(x, (n-1)k\big)$. We transform the result (by using (11), (13) and equation (8)) so as to get

$$(27) \qquad (u_t, \bar{\varphi})_n - \nu(u_x, \bar{\varphi}_x)_n + \tfrac{1}{2}\Big(\sum_i u^i(u_i + u_{\bar{i}}), \bar{\varphi}\Big)_n - (f, \bar{\varphi})_n = R^*(n),$$

where $u_t = k^{-1}(u - \bar{u})$ and $R^*(n)$ is a sum of terms of the following four different forms:

$$(\tilde{u} - \tilde{v}, \tilde{\varphi}_{i\bar{i}}), \qquad \big((\tilde{u} - \tilde{v})\tilde{w}_i, \tilde{\varphi}\big), \qquad h(\tilde{u}_i\tilde{v}_j, \tilde{\varphi}), \qquad \big((\tilde{u} - \tilde{v})\tilde{w}, \tilde{\varphi}_i\big).$$

In these expressions, $\tilde{u}$ and $\tilde{w}$ have their previous meaning (as in formula (23)), $\tilde{v}$, similarly as $\tilde{u}$, denotes any component of $\bar{u}, v, w, u$, and $\tilde{\varphi}$-any component of $\bar{\varphi}$ or its h-shifting. Note that $\tilde{u} - \tilde{v}$ in each term above denotes the difference of the same components. All functions appearing in $R^*(n)$ are taken at the same $t = nk$. The number as well as the form of $R^*(n)$ is independent of n.

Φ being fixed, we have in q the inequality

$$\varphi^2 + (\varphi_i)^2 + (\varphi_{i\bar{j}})^2 \leqslant C \quad \text{for} \quad i, j = 1, 2, 3,$$

with C independent of h. Making use of (21) and of the last inequality we may estimate $R^*(n)$ by

$$|R^*(n)| \leqslant C\{S(n)+S(n)D(n)+hD^2(n)\}, \tag{28}$$

with C independent of n. Summing up the last inequalities and then applying Cauchy's inequality, we get the estimate

$$k\Big|\sum R^*(n)\Big|$$
$$\leqslant Ck^{1/2}\Big[\Big(kN\sum S^2(n)\Big)^{1/2}+\Big(\sum S^2(n)\Big)^{1/2}\Big(k\sum D^2(n)\Big)^{1/2}+h\Big(k\sum D^2(n)\Big)^{1/2}\Big],$$

where all summations are taken over $1 \leqslant n \leqslant N$. Hence, due to (21), we get

$$k\Big|\sum R^*(n)\Big| \leqslant Ck^{1/2} = C'h \tag{29}$$

with C' independent of n.

Let us sum equations (27) over n, $1 \leqslant n \leqslant N$, and then apply the identity

$$\sum (u-\bar{u}, \bar{\varphi})_n = -k\sum (u, \varphi_t)_n - (u, \bar{\varphi})_n$$

where both summations are taken over $1 \leqslant n \leqslant N$. The result, when using the approximations U and Φ, may be written in the form

$$\int\limits_k^{Nk} dt \int\limits_\Omega \Big\{U\Phi_t - \nu\sum_i U_i\bar{\Phi}_i - \sum_i U^i U_i\bar{\Phi} + F\bar{\Phi}\Big\}\,dx + \int\limits_\Omega A^{(m)}(x)\Phi(x, 0)\,dx$$
$$= k\sum_{1\leqslant n\leqslant N} R^*(n).$$

It suffices now to pass in the last identity to the limit $h \to 0$, $h \in \{h\}$, and take into consideration (29) and the above established strong and weak convergences of U and U_i, resp., to get the desired identity (2).

11. It is clear that the construction of u is independent of the number of dimensions of x-space. However, in the two-dimensional case, due to the unicity of the weak solution of the problem (1) (see [5] or [6]), our theorem may be put in a little stronger form:

THEOREM 2. *In the two-dimensional case the original sequence* $\{U\}$ *itself converges to a weak solution of problem* (1) *if only assumption* (17) *is satisfied.*

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