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ON A RELATION BETWEEN SOME SPECIAL METHODS OF SUMMATION

BY

J. MEDER AND Z. ZDROJEWSKI (SZCZECIN)

1. S. Kaczmarz has proved a lemma (see [4], Lemma 5, p. 111) concerning a relation between the arithmetic means of numerical sequences. Meder [5] has extended this lemma (under more restrictive assumptions) to the case of logarithmic means. He has raised the problem (see [5], P 472, p. 253) of generalizing this lemma by proving it in the case of some special methods of summation. In the present note, we shall deal with a solution of this problem, extending the Kaczmarz lemma to the case of weighted means and to the case of Nörlund method of summation.

First we give some notation, definitions and statements. Let us consider a linear transformation of the form

$$T_n = \sum_{k=0}^{n} a_{nk} s_k .$$

We say that the matrix $A = (a_{nk})$ has the *mean value property* if we have an estimation of the form

$$\mathrm{M(A)} \qquad \Big|\sum_{v=0}^{k} a_{nv} s_v\Big| \leqslant C \Big|\sum_{v=0}^{m} a_{mv} s_v\Big| \quad \text{for} \quad 0 \leqslant m \leqslant k \leqslant n,$$

where C is an absolute constant (see [7], Lemma 1, p. 50).

In the sequel we shall need the following lemma of W. Jurkat and A. Peyerimhoff:

If the matrix $A = (a_{nk})$ satisfies the conditions

$$\text{(a)} \qquad a_{kv} \neq 0, \quad 0 < \frac{a_{nv}}{a_{kv}} \leqslant C \quad (0 \leqslant v \leqslant k \leqslant n),$$

and

$$\text{(b)} \qquad \frac{a_{nv}}{a_{kv}} \searrow \quad \text{for} \quad v \nearrow \quad (0 \leqslant v \leqslant k \leqslant n),$$

then for an arbitrary sequence $\{s_n\}$ we have the inequality

$$\text{(N)} \qquad \Big|\sum_{v=0}^{k} a_{nv} s_v\Big| \leqslant \frac{a_{n0}}{a_{m0}} \Big|\sum_{v=0}^{m} a_{mv} s_v\Big| \qquad (0 \leqslant m \leqslant k \leqslant n)$$

(see [3], Lemma 1, p. 153, and Vergleichssätze, p. 173).

The mean value property follows then immediately from inequality (N) and condition (a).

Below we apply this result to the Nörlund method of summability.

We suppose that $p_0 > 0$, $p_n \geqslant 0$, $P_n = p_0 + p_1 + \ldots + p_n$, and define t_n by

$$t_n = \frac{p_n s_0 + p_{n-1} s_1 + \ldots + p_0 s_n}{P_n}.$$

If $t_n \to s$ when $n \to \infty$, and $s_n = a_0 + a_1 + \ldots + a_n$, we shall write (see [2], p. 64)

$$(N, p_n) - \lim s_n = s \qquad \text{or} \qquad (N, p_n) - \sum a_n = s.$$

Then we call t_n the *n-th (N, p_n)-mean* of the sequence $\{s_n\}$.

In general, the (N, p_n)-means do not satisfy the mean value theorem. We have, however, the following

LEMMA A. *Let $0 < p_n \searrow$ and $p_{n+1}/p_n \nearrow$ for $n \nearrow$. Then we have*

$$\Big|\sum_{v=0}^{k} p_{n-v} s_v\Big| \leqslant P_m |t_m| \quad \textit{for} \quad 0 \leqslant m \leqslant k \leqslant n.$$

This lemma follows immediately from the above quoted Lemma of W. Jurkat and A. Peyerimhoff.

In order to formulate the next lemma, we introduce the following class of (N, p_n)-means, to be denoted by M^*:

A sequence $\{p_n\}$ is said to belong to M^* if

$$\text{(i)} \qquad p_n > 0 \quad (n = 0, 1, \ldots),$$

$$\text{(j)} \qquad \{p_n\} \text{ is convex or concave,}$$

$$\text{(k)} \qquad 0 < \liminf_{n\to\infty} (n+1) p_n / P_n \leqslant \limsup_{n\to\infty} (n+1) p_n / P_n < \infty$$

(see [6], p. 244).

LEMMA B. *If $\{p_n\} \in M^*$, then the sequence $\{n(p_n - p_{n-1})/p_n\}$ is bounded* (see [6], Lemma 2, p. 244).

Remark. If $\{p_n\} \in M^*$, being a decreasing and convex sequence, the boundedness of the sequence $\{n(p_n - p_{n-1})/p_n\}$ follows immediately from a lemma of L. Mc. Fadden (see [1], Lemma (5.13)). Lemma 2 is also true if we assume only that $p_0 > 0$ and $p_n \geqslant 0$ for $n = 1, 2, \ldots$, provided that $\{p_n\}$ is concave (see [6], Remark, p. 247).

We introduce one more class of (N, p_n)-means, to be denoted by M: A sequence $\{p_n\}$ is said to belong to M if

(A) $\quad 0 < p_n \searrow, \; p_{n+1}/p_n \nearrow \quad$ for $\quad n \nearrow$ and $\liminf\limits_{n\to\infty} (n+1)p_n/P_n > 0$

or

(B) $\quad 0 < p_n \nearrow, \; p_{n+1}/p_n \searrow \quad$ for $\quad n \nearrow$ and $\limsup\limits_{n\to\infty} (n+1)p_n/P_n < \infty$.

2. We are now going to prove a theorem which is an answer to the above mentioned problem P 472 in that it extends the lemma spoken of to the case of (N, p_n)-means with $\{p_n\}$ fulfilling condition (A). Namely we have

THEOREM 1. *Let there be given an increasing sequence of positive numbers λ_n and a series $\sum\limits_{n=0}^{\infty} a_n$.*

Denote by t_n, T_n and $T_n^{(1)}$ the (N, p_n)-means of the series

$$\sum_{n=0}^{\infty} a_n, \qquad \sum_{n=0}^{\infty} \frac{a_n}{\lambda_n} \qquad \text{and} \qquad \sum_{n=0}^{\infty} S_n^{(1)} \Delta^2 \frac{1}{\lambda_n},$$

respectively. Here $S_n^{(1)} = s_0 + s_1 + \ldots + s_n$, where $s_n = a_0 + a_1 + \ldots + a_n$.

Further assume that

(a) $$\{p_n\} \text{ satisfies condition (A)},$$

(b) $$t_n = o(\lambda_n),$$

(c) $$\Delta \frac{1}{\lambda_n} = O\left[\frac{1}{(n+1)\lambda_n}\right].$$

Then we have

$$T_n = T_n^{(1)} + o(1) \quad \text{for} \quad n \to \infty.$$

Proof. Let $\{p_n\}$ satisfy condition (a). It is then convex and, moreover $\{p_n\} \in M^*$. Therefore, according to Lemma B, we have the relation

(1) $$\frac{n(p_n - p_{n-1})}{p_n} = O(1).$$

The definitions of t_n, T_n and $T_n^{(1)}$ imply

$$t_n = \frac{1}{P_n} \sum_{k=0}^{n} a_k \sum_{v=k}^{n} p_{n-v},$$

$$T_n = \frac{1}{P_n} \sum_{k=0}^{n} \frac{a_k}{\lambda_k} \sum_{v=k}^{n} p_{n-v},$$

$$T_n^{(1)} = \frac{1}{P_n} \sum_{k=0}^{n} S_k^{(1)} \Delta^2 \frac{1}{\lambda_k} \sum_{v=k}^{n} p_{n-v}.$$

We can write

$$T_n = \frac{t_n}{\lambda_n} + \frac{1}{P_n} \sum_{k=0}^{n} a_k \left(\frac{1}{\lambda_k} - \frac{1}{\lambda_n}\right) \sum_{v=k}^{n} p_{n-v}. \tag{2}$$

Let us write

$$\eta_k = \begin{cases} \left(\frac{1}{\lambda_k} - \frac{1}{\lambda_n}\right) \sum_{v=k}^{n} p_{n-v} & \text{for} \quad k = 0, 1, 2, \ldots, n, \\ 0 & \text{for} \quad k = n+1, n+2, \ldots \end{cases} \tag{3}$$

Then we have for $k \leqslant n$ the equations

$$\Delta \eta_k = p_{n-k}\left(\frac{1}{\lambda_k} - \frac{1}{\lambda_n}\right) + \Delta \frac{1}{\lambda_k} \sum_{v=k+1}^{n} p_{n-v} \tag{4}$$

and

$$\Delta^2 \eta_k = \Delta^2 \frac{1}{\lambda_k} \sum_{v=k+1}^{n} p_{n-v} + \Delta \frac{p_{n-k}}{\lambda_k} + p_{n-k-1} \Delta \frac{1}{\lambda_{k+1}} - \frac{1}{\lambda_n} \Delta p_{n-k}. \tag{5}$$

Owing to (3), (4) and to the equations $\eta_n = \Delta \eta_n = \Delta^2 \eta_n = 0$, we can write formula (2) in the form

$$T_n = \frac{t_n}{\lambda_n} + \frac{1}{P_n} \sum_{k=0}^{n} S_k^{(1)} \Delta^2 \eta_k. \tag{6}$$

If we put here the value given by formula (5) in place of $\Delta^2 \eta_k$, we obtain

$$\begin{aligned} T_n = {} & \left[\frac{t_n}{\lambda_n} - \frac{1}{P_n \lambda_n} \sum_{k=0}^{n} S_k^{(1)} \Delta p_{n-k}\right] + \frac{1}{P_n} \sum_{k=0}^{n} S_k^{(1)} \Delta \frac{p_{n-k}}{\lambda_k} + \\ & + \frac{1}{P_n} \sum_{k=0}^{n} S_k^{(1)} p_{n-k-1} \Delta \frac{1}{\lambda_{k+1}} + \frac{1}{P_n} \sum_{k=0}^{n} S_k^{(1)} \Delta^2 \frac{1}{\lambda_k} \sum_{v=k+1}^{n} p_{n-v} \\ = {} & S_1 + S_2 + S_3 + S_4, \qquad \text{say.} \end{aligned} \tag{7}$$

We pass now to the estimation of S_1, S_2, S_3 and S_4. We have

$$S_1 = \frac{t_n}{\lambda_n} - \frac{1}{P_n \lambda_n} \sum_{k=0}^{n} p_{n-k} s_k = 0 \tag{8}$$

and

$$S_2 = \frac{1}{P_n}\sum_{k=0}^{n} S_k^{(1)} \Delta \frac{p_{n-k}}{\lambda_k} = \frac{1}{P_n}\sum_{k=0}^{n} \frac{p_{n-k}s_k}{\lambda_k}$$

$$= \frac{t_n}{\lambda_n} + \frac{1}{P_n}\sum_{k=0}^{n}\Big(\sum_{v=0}^{k} p_{n-v}s_v\Big)\Delta\frac{1}{\lambda_k} \qquad (\lambda_{-1} = 0).$$

Note that by virtue of the last of conditions (A) we have

(9) $$\frac{1}{P_n}\sum_{k=0}^{n}\frac{P_k}{k+1} = O(1).$$

In fact, let $\liminf (n+1)p_n/P_n = \alpha > 0$, and let ε be an arbitrarily small positive number less than α. We have $(k+1)p_k/P_k > \alpha - \varepsilon > 0$ for $k > k_0$. Hence, for $k > k_0$,

$$\frac{1}{P_n}\sum_{k=0}^{n}\frac{P_k}{k+1} = \frac{1}{P_n}\sum_{k=0}^{k_0}\frac{P_k}{k+1} + \frac{1}{P_n}\sum_{k=k_0+1}^{n}\frac{p_k}{\alpha-\varepsilon} = O(1).$$

Now, by virtue of the assumptions of Lemma 1 and according to Lemma A, we have

$$|S_2| < \frac{|t_n|}{\lambda_n} + \frac{1}{P_n}\sum_{k=0}^{n} P_m|t_m| O\Big[\frac{1}{(k+1)\lambda_k}\Big] \quad \text{for} \quad m \leqslant k.$$

Hence

$$S_2 = o(1) + \frac{1}{P_n}\sum_{k=0}^{n} P_k o(\lambda_k) O\Big[\frac{1}{(k+1)\lambda_k}\Big] = o(1) + o(1)\frac{1}{P_n}\sum_{k=0}^{n}\frac{P_k}{k+1},$$

and by (9) we obtain

(10) $$S_2 = o(1).$$

Next we have

$$S_3 = \frac{1}{P_n}\sum_{k=0}^{n-1} p_{n-k-1}s_k \frac{1}{p_{n-k-1}}\sum_{v=k}^{n-1} p_{n-v-1}\Delta\frac{1}{\lambda_{v+1}}$$

$$= \frac{P_{n-1}}{P_n} t_{n-1}\Delta\frac{1}{\lambda_n} + \frac{1}{P_n}\sum_{k=0}^{n-2}\Big(\sum_{v=0}^{k} p_{n-v-1}s_v\Big)\Delta\Big[\frac{1}{p_{n-k-1}}\sum_{v=k}^{n-1} p_{n-v-1}\Delta\frac{1}{\lambda_{v+1}}\Big].$$

By virtue of Lemma A and Lemma B, and in view of the assumptions of Theorem 1, we conclude that

$$\Delta[\;] = \frac{(n-k)(p_{n-k-2}-p_{n-k-1})}{p_{n-k-2}}\,\frac{P_{n-k-1}}{(n-k)p_{n-k-1}}\,O\left[\frac{1}{(k+1)\lambda_k}\right]+$$

$$+\frac{p_{n-k-1}}{p_{n-k-2}}\,O\left[\frac{1}{(k+1)\lambda_k}\right] = O\left[\frac{1}{(k+1)\lambda_k}\right].$$

Therefore

$$S_3 = o(1)+\frac{1}{P_n}\sum_{k=0}^{n} P_k o(\lambda_k)\,O\left[\frac{1}{(k+1)\lambda_k}\right] = o(1)$$

or

(11) $$S_3 = o(1).$$

Passing to the estimation of S_4, let us note that

$$S_4 = T_n^{(1)} - \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k} S_k^{(1)} \Delta^2\frac{1}{\lambda_k} = T_n^{(1)} - A, \quad \text{say}.$$

Now

$$A = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k} S_k^{(1)}\Delta^2\frac{1}{\lambda_k} = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}s_k\frac{1}{p_{n-k}}\sum_{v=0}^{n} p_{n-v}\Delta^2\frac{1}{\lambda_v}$$

$$= t_n\Delta^2\frac{1}{\lambda_n}+\frac{1}{P_n}\sum_{k=0}^{n-1}\left(\sum_{v=0}^{k} p_{n-v}s_v\right)\Delta\left[\frac{1}{p_{n-k}}\sum_{v=k}^{n} p_{n-v}\Delta^2\frac{1}{\lambda_v}\right].$$

Since

$$\Delta[\;] = \Delta\left(\frac{1}{p_{n-k}}\right)\sum_{v=k+1}^{n} p_{n-v}\Delta^2\frac{1}{\lambda_k}+\Delta^2\frac{1}{\lambda_k},$$

we have, by virtue of Lemma B and in view of conditions (a) and (c),

$$\Delta[\;] = O\left[\frac{1}{(k+1)\lambda_k}\right].$$

Because of the last relation in the formula for A and of Lemma A, condition (b) implies that

$$A = o(1)+\frac{1}{P_n}\sum_{k=0}^{n} P_k o(\lambda_k)\,O\left[\frac{1}{(k+1)\lambda_k}\right] = o(1).$$

Hence

(12) $$S_4 = T_n^{(1)} + o(1).$$

Combining equations (8), (10), (11) and (12), we obtain the required relation.

3. In this section we shall generalize the Kaczmarz lemma to the case of weighted means.

If $p_n \geqslant 0$, $p_0 > 0$, $\sum p_n = \infty$ (so that $P_n = p_0 + p_1 + \ldots + p_n \to \infty$), and

$$\bar{t}_n = \frac{p_0 s_0 + p_1 s_1 + \ldots + p_n s_n}{p_0 + p_1 + \ldots + p_n} \to s$$

when $n \to \infty$, then we say that $s_n \to s(\overline{N}, p_n)$ (see [2], p. 57).

We shall call $\bar{t}_n$ the *n-th* $(\overline{N}, p_n)$*-mean* or the *weighted mean* of the sequence $\{s_n\}$.

The method $(\overline{N}, p_n > 0)$ satisfies the conditions of the mean value theorem.

THEOREM 2. *Let there be given an increasing sequence of positive numbers* λ_n *and a series* $\sum_{n=0}^{\infty} a_n$.

Denote by $\bar{t}_n$, $\overline{T}_n$ *and* $\overline{T}_n^{(1)}$ *the* $(\overline{N}, p_n)$*-means of the series*

$$\sum_{n=0}^{\infty} a_n, \quad \sum_{n=0}^{\infty} \frac{a_n}{\lambda_n} \quad \text{and} \quad \sum_{n=0}^{\infty} S_n^{(1)} \Delta^2 \frac{1}{\lambda_n},$$

respectively, where $S_n^{(1)} = s_0 + s_1 + \ldots + s_n$, *and* $s_n = a_0 + a_1 + \ldots + a_n$.

If

1) $\{p_n\} \in M^*$,

2) $\bar{t}_n = o(\lambda_n)$,

3) $\Delta \dfrac{1}{\lambda_n} = O\left[\dfrac{1}{(n+1)\lambda_n}\right]$,

then

$$\overline{T}_n = \overline{T}_n^{(1)} + o(1).$$

Proof. The definitions of $\bar{t}_n$, $\overline{T}_n$ and $\overline{T}_n^{(1)}$ imply that

$$\bar{t}_n = \frac{1}{P_n} \sum_{k=0}^{n} a_k \sum_{v=k}^{n} p_v,$$

$$\overline{T}_n = \frac{1}{P_n} \sum_{k=0}^{n} \frac{a_k}{\lambda_k} \sum_{v=k}^{n} p_v$$

and

$$\overline{T}_n^{(1)} = \frac{1}{P_n}\sum_{k=0}^{n} S_k^{(1)} \Delta^2 \frac{1}{\lambda_k}\sum_{v=k}^{n} p_v .$$

We can write $\overline{T}_n$ in the form

$$\overline{T}_n = \frac{\bar{t}_n}{\lambda_n} + \frac{1}{P_n}\sum_{k=0}^{n} a_k\left(\frac{1}{\lambda_k}-\frac{1}{\lambda_n}\right)\sum_{v=k}^{n} p_v . \tag{13}$$

Let us write

$$\bar{\eta}_k = \begin{cases} \left(\dfrac{1}{\lambda_k}-\dfrac{1}{\lambda_n}\right)\displaystyle\sum_{v=k}^{n} p_v & \text{for} \quad k = 1, 2, \ldots, n, \\ 0 & \text{for} \quad k = n+1, n+2, \ldots \end{cases} \tag{14}$$

Then for $k \leqslant n$ we have

$$\Delta\bar{\eta}_k = \Delta\frac{1}{\lambda_k}\sum_{v=k+1}^{n} p_v + \left(\frac{1}{\lambda_k}-\frac{1}{\lambda_n}\right)p_k \tag{15}$$

and

$$\Delta^2\bar{\eta}_k = \Delta^2\frac{1}{\lambda_k}\sum_{v=k+1}^{n} p_v + p_{k+1}\Delta\frac{1}{\lambda_{k+1}} - \Delta\left(\frac{p_k}{\lambda_k}\right) - \frac{1}{\lambda_n}\Delta p_k . \tag{16}$$

Because of (14), (15), (16) and $\eta_n = \Delta\eta_n = \Delta^2\eta_n = 0$, we can rewrite formula (13) in the form

$$\overline{T}_n = \frac{\bar{t}_n}{\lambda_n} + \frac{1}{P_n}\sum_{k=0}^{n} S_k^{(1)}\Delta^2\bar{\eta}_k . \tag{17}$$

Replacing $\Delta^2\bar{\eta}_k$ in (17) by the right-hand side of (16), we obtain

$$\begin{aligned} \overline{T}_n &= \left[\frac{t_n}{\lambda_n} - \frac{1}{P_n}\sum_{k=0}^{n} S_k^{(1)}\Delta\left(\frac{p_k}{\lambda_k}\right)\right] - \frac{1}{nP_n}\sum_{k=0}^{n} S_k^{(1)}\Delta p_k + \\ &\quad + \frac{1}{P_n}\sum_{k=0}^{n} p_{k+1} S_k^{(1)}\Delta\frac{1}{\lambda_{k+1}} + \frac{1}{P_n}\sum_{k=0}^{n} S_k^{(1)}\Delta^2\frac{1}{\lambda_k}\sum_{v=k+1}^{n} p_v \\ &= I_1 + I_2 + I_3 + I_4, \qquad \text{say.} \end{aligned} \tag{18}$$

When estimating $\bar{T}_n$, we shall consider the following two cases: 1° $0 < p_n \searrow$ and $\{p_n\}$ is convex, 2° $0 < p_n \nearrow$ and $\{p_n\}$ is concave or convex. In case 1° we have

$$I_1 = \frac{\bar{t}_n}{\lambda_n} - \frac{1}{P_n}\sum_{k=0}^{n} s_k \sum_{v=k}^{n}\left(\frac{p_v}{\lambda_v} - \frac{p_{v+1}}{\lambda_{v+1}}\right)$$

$$= \frac{\bar{t}_n}{\lambda_n} - \frac{1}{P_n}\sum_{k=0}^{n}\frac{p_k s_k}{\lambda_k} + \frac{p_{n+1}}{P_n\lambda_{n+1}}\sum_{k=0}^{n} s_k$$

$$= -\frac{1}{P_n}\sum_{k=0}^{n-1}\left(\sum_{v=0}^{k} p_v s_v\right)\Delta\frac{1}{\lambda_k} + \frac{p_{n+1}}{P_n\lambda_{n+1}}\sum_{k=0}^{n} p_k s_k \frac{1}{p_k} = B_1 + B_2, \qquad \text{say.}$$

By the assumptions of Theorem 2 we have

$$B_1 = \frac{1}{P_n}\sum_{k=0}^{n} P_k o(\lambda_k) O\left[\frac{1}{(k+1)\lambda_k}\right] = o(1).$$

Similarly

$$|B_2| \leqslant \frac{p_{n+1}}{p_n P_n \lambda_{n+1}}\left|\sum_{k=0}^{n} p_k s_k\right| + \frac{p_{n+1}}{P_n\lambda_{n+1}}\sum_{k=0}^{n-1} P_k o(\lambda_k)\left|\Delta\frac{1}{p_k}\right|$$

$$= o(1) + \frac{p_{n+1}}{P_n}\sum_{k=0}^{n}\frac{P_k}{(k+1)p_k} o(\lambda_k)\frac{1}{\lambda_k}\frac{(k+1)(p_k - p_{k+1})}{p_{k+1}}$$

$$= o(1) + \frac{p_{n+1}}{P_n}\sum_{k=0}^{n} o(1) = o(1) + \frac{(n+1)p_n}{P_n} o(1) = o(1) + o(1) = o(1).$$

Hence $B_2 = o(1)$ which together with $B_1 = o(1)$ gives the relation

$$I_1 = o(1). \tag{19}$$

In a similar manner we obtain

$$I_2 = o(1). \tag{20}$$

Now

$$I_3 = \frac{1}{P_n}\sum_{k=0}^{n} p_k s_k \frac{1}{p_k}\sum_{v=k}^{n} p_{v+1}\Delta\frac{1}{\lambda_v}$$

$$= \frac{p_{n+1}}{p_n}\bar{t}_n\Delta\frac{1}{\lambda_n} + \frac{1}{P_n}\sum_{k=0}^{n-1}\left(\sum_{v=0}^{k} p_v s_v\right)\Delta\left[\frac{1}{p_k}\sum_{v=k}^{n} p_{v+1}\Delta\frac{1}{\lambda_v}\right]$$

where

$$\Delta[\;] = \Delta\frac{1}{p_k}\sum_{v=k}^{n} p_{v+1}\Delta\frac{1}{\lambda_v} + \Delta\frac{1}{\lambda_k}.$$

Taking into consideration the last formula of the right-hand side of the formula for I_3, and applying Lemma B, we obtain under the assumptions of Theorem 2 the estimation

$$|I_3| < \frac{|\bar{t}_n|}{\lambda_n} + \frac{1}{P_n}\sum_{k=0}^{n}\frac{P_k}{k+1}o(\lambda_k)\frac{1}{\lambda_k}\frac{(k+1)(p_k-p_{k+1})}{p_k} +$$
$$+\frac{1}{P_n}\sum_{k=0}^{n}P_k o(\lambda_k) O\left[\frac{1}{(k+1)\lambda_k}\right] = o(1).$$

Hence

(21) $$I_3 = o(1).$$

Finally, we have

$$I_4 = \overline{T}_n^{(1)} - \frac{1}{P_n}\sum_{k=0}^{n} p_k S_k^{(1)}\Delta^2\frac{1}{\lambda_k} = \overline{T}_n^{(1)} - B_3, \qquad \text{say},$$

where

$$B_3 = \frac{1}{P_n}\sum_{k=0}^{n} p_k s_k \frac{1}{p_k}\sum_{v=k}^{n} p_v\Delta^2\frac{1}{\lambda_v}$$
$$= \bar{t}_n\Delta^2\frac{1}{\lambda_n} + \frac{1}{P_n}\sum_{k=0}^{n}\left(\sum_{v=0}^{k} p_v s_v\right)\Delta\left[\frac{1}{p_k}\sum_{v=k}^{n} p_v\Delta^2\frac{1}{\lambda_v}\right].$$

Arguing as above we conclude that

$$\Delta[\;] = O\left[\frac{1}{(k+1)\lambda_k}\right],$$

which implies $B_3 = o(1)$. Thus we have

(22) $$I_4 = \overline{T}_n^{(1)} + o(1).$$

By virtue of (19), (20), (21) and (18), we obtain the required relation in the examined case.

In case 2°, where $\{p_n\}$ is increasing, we have to check that the sequence $\{p_{n+1}/p_n\}$ is bounded. As a matter of fact, if $\{p_n\}$ is concave, then

$$p_{n+1} - p_n < p_n - p_{n-1} < p_1 - p_0,$$

whence it follows that

$$\frac{p_{n+1}}{p_n} = O(1) \quad \text{as} \quad n \to \infty.$$

If $\{p_n\}$ is convex, then assuming $\{p_{n+1}/p_n\}$ to be unbounded, we should find for arbitrarily large positive number C_1 an increasing sequence $\{n_k\}$ of indices such that

$$\frac{p_{n_k+1}}{p_{n_k}} > C_1 + 1 \quad \text{for} \quad k > k_0.$$

Hence we would have $p_{n_k+1} - p_{n_k} > C_1 p_{n_k}$ for $k > k_0$ and

$$\frac{(n_k+1)(p_{n_k+1} - p_{n_k})}{p_{n_k+1}} > C_1 \frac{(n_k+1)p_{n_k}}{p_{n_k+1}} > C_1 \frac{P_{n_k}}{P_{n_k+1}}$$
$$= C_1 \frac{P_{n_k+1} - p_{n_k+1}}{P_{n_k+1}} = C_1\left(1 - \frac{p_{n_k+1}}{P_{n_k+1}}\right).$$

Putting

$$\limsup_{n\to\infty} \frac{(n+1)p_n}{P_n} = \beta,$$

we obtain from the last inequalities that $\beta \geqslant C_1$, which by the very meaning of C_1 contradicts $\{p_n\} \in M^*$. Thus $\{p_{n+1}/p_n\}$ is bounded in both cases.

Except for some small differences, the estimations take now a similar course as in the case of a decreasing $\{p_n\}$. Therefore we confine ourselves to indicate only the estimate of B_3. We have

$$B_3 = \bar{t}_n \Delta^2 \frac{1}{\lambda_n} + \frac{1}{P_n} \sum_{k=0}^{n} \left(\sum_{v=0}^{k} p_v s_v\right) \Delta\left[\frac{1}{p_k} \sum_{v=k}^{n} p_v \Delta^2 \frac{1}{\lambda_v}\right].$$

Since

$$\Delta[\ ] = \frac{p_{k+1} - p_k}{p_k p_{k+1}} \sum_{v=k}^{n} p_v \Delta^2 \frac{1}{\lambda_v} + \frac{p_k}{p_{k+1}} \Delta^2 \frac{1}{\lambda_k},$$

we have

$$|\Delta[\ ]| < 2p_n \frac{p_{k+1} - p_k}{p_{k+1}} \frac{1}{p_k} \frac{1}{\lambda_k} + \Delta^2 \frac{1}{\lambda_k},$$

and, consequently,

$$|B_3| < \frac{|\bar{t}_n|}{\lambda_n} + \frac{2p_n}{P_n} \sum_{k=0}^{n} \frac{P_k}{(k+1)p_k} o(\lambda_k) \frac{1}{\lambda_k} \frac{(k+1)(p_{k+1}-p_k)}{p_{k+1}} +$$

$$+ \frac{1}{P_n} \sum_{k=0}^{n} P_k o(\lambda_k) O\left[\frac{1}{(k+1)\lambda_k}\right]$$

$$= o(1) + \frac{p_n}{P_n} \sum_{k=0}^{n} o(1) + \frac{o(1)}{P_n} \sum_{k=0}^{n} p_k = o(1) + O(1)o(1) = o(1).$$

Therefore $B_3 = o(1)$, which together with the above remark ends the proof of Theorem 2.

At the end it seems to us worth while to put the problem of extension of Theorem 1 to the case when $0 < p_n \nearrow$ (**P 632**). This problem seems to be more difficult because for increasing $\{p_n\}$ the mean value theorem is not applicable. It would be also of interest to generalize the obtained results to more general classes of special methods of summation.

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