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EIGENVALUES OF INFINITE MATRICES

BY

H. HANANI, E. NETANYAHU AND M. REICHAW (HAIFA)

Introduction. Let X and Y be linear spaces whose points are sequences of complex numbers denoted by $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ respectively. Let $A\colon X \to Y$ be a linear operator (not necessarily continuous) of X into Y represented by an infinite matrix $A = (a_{ij})$, $i, j = 1, 2, \ldots$, where a_{ij} are complex numbers and $y = Ax = (y_1, y_2, \ldots)$ is given by

$$y_i = \sum_{j=1}^{\infty} a_{ij} x_j. \tag{a}$$

If A is a matrix and X is a set such that there exists a point $x \neq 0$ of X and a number λ satisfying $Ax = \lambda x$, then λ will be called an *eigenvalue of the matrix A relative to the set X* (1). The point x will be called an *eigenvector* of the matrix A belonging to λ.

Section 1 of this paper starts with an example of a matrix A and of two Banach spaces such that $\lambda = 0$ is an eigenvalue of A relative to one space and is not an eigenvalue of A relative to the second space. This example has the additional property that the operator A defined by a matrix (a_{ij}), $i, j = 1, 2, \ldots$, satisfies a condition used by S. Gersgorin for finite matrices. The trivial example of the shift operator having eigenvalue 1 on m but not on l does not have this additional property. Theorems 1 and 2 of this section contain conditions under which the number $\lambda = 0$ is not an eigenvalue of a matrix A relative to some spaces. Their application to the operator $A - \lambda I$, where I is the identity operator on the space X, gives estimates for eigenvalues of the matrix A relative to X.

The Theorems proved in section 2 are related to a known theorem of Gersgorin (see [1]) from the theory of finite matrices and their proof is similar to that of Theorem 1 in [3].

In section 3 a known theorem about the existence of an inverse

(1) We do not assume that A is defined on the whole of X.

operator in general Banach spaces is used to obtain various conditions for an operator $A: X \to Y$, defined by (a), to be one-to-one. In particular, the mentioned theorem of Gersgorin stating that for a finite $\nu \times \nu$ matrix $A = (a_{ij})$ one has $\det A \neq 0$ provided that

$$\text{(b)} \qquad |a_{ii}| > \sum_{j \neq i} |a_{ij}|, \qquad i = 1, 2, \ldots, \nu,$$

is generalized to the case of infinite matrices. Since the theorems proved in sections 1-3 are dealing with infinite matrices, one can indicate some applications of the obtained results in quantum mechanics. This is done in section 4 where the harmonic and anharmonic oscilators are considered. Let us finally note that the results in Theorems 1 and 2 cannot be obtained by using known theorems in functional analysis (using methods of functional analysis one can only obtain a result similar to that in Theorem 2 if we assume the strong inequality $<$ in (b_5)). In the sequel, we denote by m ($\tilde{m}$) the Banach space of all bounded sequences $x = (x_1, x_2, \ldots)$ of real (complex) numbers x_i, $i = 1, 2, \ldots$, with norm

$$\|x\| = \sup_{1 \leqslant i < \infty} |x_i|$$

and by l^p ($\tilde{l}^p$) the Banach space of all sequences $x = (x_1, x_2, \ldots)$ of real (complex) numbers x_i, for which $\sum_{i=1}^{\infty} |x_i|^p < \infty$, with norm

$$\|x\| = \Big(\sum_{i=1}^{\infty} |x_i|^p\Big)^{1/p}.$$

By m_ν ($\tilde{m}_\nu$), l_ν^p ($\tilde{l}_\nu^p$) we denote spaces of *finite* sequences $x = (x_1, x_2, \ldots \ldots, x_\nu)$ of real (complex) numbers x_i, $i = 1, 2, \ldots, \nu$, and norms

$$\|x\| = \max_{1 \leqslant i \leqslant \nu} |x_i| \qquad \text{for} \qquad m_\nu \ (\tilde{m}_\nu)$$

and

$$\|x\| = \Big(\sum_{i=1}^{\nu} |x_i|^p\Big) \qquad \text{for} \qquad l_\nu^p \ (\tilde{l}_\nu^p).$$

We write also l ($\tilde{l}$), l_ν ($\tilde{l}_\nu$) instead of l^1 ($\tilde{l}^1$), $\tilde{l}_\nu^1$ ($\tilde{l}_\nu^1$).

1. Let us begin with an example of a matrix $A = (a_{ij})$ such that $\lambda = 0$ is an eigenvalue of A relative to the space m and $\lambda = 0$ is not an eigenvalue of A relative to the space $\tilde{l}$. Incidentally the same example will show that for infinite matrices $A = (a_{ij})$, $i, j = 1, 2, \ldots$, the condition

$$(\text{b}_1) \qquad |a_{ii}| > \sum_{j \neq i} |a_{ij}|, \qquad i = 1, 2, \ldots$$

(similar to (b)) does not imply in general that the operator $A: X \to Y$, defined by (a), is one-to-one.

Example 1. Let $A = (a_{ij})$, $i, j = 1, 2, \ldots$, be an infinite matrix defined by $a_{ii} = 1$, $a_{i,i+1} = 1 - 1/2^i$ and $a_{ij} = 0$ for $j \neq i$ and $j \neq i+1$, and let $X = Y = m$. Then the operator $A: X \to X$ defined by (a) is, as easily seen, a linear (even continuous) operator of X into itself. The matrix $A = (a_{ij})$ satisfies also (b_1). However $A: X \to X$ is not one-to-one. Indeed, for an arbitrary $x_1 \neq 0$ the sequence of equations

$$x_n + \left(1 - \frac{1}{2^n}\right) x_{n+1} = 0$$

has a non-zero solution $x = (x_1, x_2, \ldots, x_n, \ldots)$ where

$$(b_2) \qquad x_n = \frac{(-1)^{n-1}}{\prod_{i=1}^{n-1}\left(1 - \frac{1}{2^i}\right)} x_1$$

and obviously $x \in X$. Thus $x = (x_1, x_2, \ldots, x_n, \ldots)$, where x_n is given by (b_2) and $x_1 \neq 0$, is an eigenvector of A belonging to the eigenvalue $\lambda = 0$ of the matrix A relative to the space $X = m$. It is a trivial consequence of the subsequent Theorem 1 (see Remark 1) that $\lambda = 0$ is not an eigenvalue of the matrix A relative to the space $X = \tilde{l}$.

Theorems 1 and 2 give sufficient conditions for a matrix A under which the number $\lambda = 0$ is not an eigenvalue of A relative to the space $\tilde{l}^q$ (l^q) for $q \geq 1$.

THEOREM 1. *Let* $A = (a_{ij})$, $i, j = 1, 2, \ldots$, *be an infinite matrix with* $a_{ii} \neq 0$ *and let*

$$(b_3) \qquad A_j = \sum_{i=1, i\neq j}^{\infty} \left|\frac{a_{ij}}{a_{ii}}\right| < 1 \quad \textit{for all } j = 1, 2, \ldots$$

Then $\lambda = 0$ *is not an eigenvalue of* A *relative to* $\tilde{l}$.

Proof. Suppose to the contrary that there exists a point $x = (x_1, x_2, \ldots) \in \tilde{l}$ such that $x \neq 0$ and $Ax = 0$. Then $\sum_{j=1}^{\infty} a_{ij} x_j = 0$ and thus

$$x_i = -\sum_{j=1, j\neq i}^{\infty} \frac{a_{ij}}{a_{ii}} x_j .$$

Hence

$$|x_i| \leq \sum_{j=1, j\neq i}^{\infty} \frac{|a_{ij}|}{|a_{ii}|} |x_j|,$$

and

$$(\mathrm{b}_4)\qquad \sum_{i=1}^{\infty}|x_i| \leqslant \sum_{i=1}^{\infty}\sum_{j=1,j\neq i}^{\infty}\left|\frac{a_{ij}}{a_{ii}}\right||x_j| = \sum_{j=1}^{\infty}|x_j|\sum_{i=1,i\neq j}^{\infty}\left|\frac{a_{ij}}{a_{ii}}\right| = \sum_{j=1}^{\infty}A_j|x_j|.$$

Since not all x_j are equal to zero and $A_j < 1$, (b_4) is impossible.

Remark 1. Since the matrix A defined in Example 1 satisfies (b_3), it follows that $\lambda = 0$ is not an eigenvalue of this matrix relative to $\tilde{l}$.

Let us note that if A satisfies (b_3) and A is considered as a linear operator defined on a subset of $\tilde{l}$ according to formula (a), then by Theorem 1 one gets that A is a one-to-one mapping on this subset.

It is also easily seen that a statement similar to that of Theorem 1 holds in the case of *finite* $\nu\times\nu$ matrices (a_{ij}), $i,j = 1,2,\ldots,\nu$ which may be considered as operators of $\tilde{l}_\nu$ into itself. Thus if $A = (a_{ij})$, $i,j = 1,2,\ldots,\nu$, is a finite $\nu\times\nu$ matrix such that

$$(\mathrm{b}_3)\qquad A_j = \sum_{i=1,i\neq j}^{\nu}\left|\frac{a_{ij}}{a_{ii}}\right| < 1 \quad \text{for} \quad j = 1,2,\ldots,\nu,$$

then $\det A \neq 0$ (2).

Example 2. Let

$$A = \begin{pmatrix} 5 & 3 & 3 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}.$$

Then $A_j < 1$ for $j = 1,2,3$. Thus $\det A \neq 0$. Let us note that condition (b) does not hold for this matrix A.

Remark 2. As a consequence of Theorem 1, we obtain that each eigenvalue of an arbitrary infinite matrix $A = (a_{ij})$, $i,j = 1,2,\ldots$, relative to the space $\tilde{l}$ satisfies one of the inequalities

$$\sum_{i=1,i\neq j}^{\infty}\left|\frac{a_{ij}}{a_{ii}-\lambda}\right| \geqslant 1, \quad j = 1,2,\ldots$$

If for some j, $a_{ij} = 0$ for $i \neq j$ one has to multiply first both sides of this inequality by $|a_{ii}-\lambda|$. Then one gets $a_{ii} = \lambda$.

Similarly, each eigenvalue of an arbitrary finite $\nu\times\nu$ matrix (a_{ij}), $i,j = 1,\ldots,\nu$, satisfies one of the inequalities

$$\sum_{i=1,i\neq j}^{\nu}\left|\frac{a_{ij}}{a_{ii}-\lambda}\right| \geqslant 1, \quad j = 1,2,\ldots,\nu.$$

(2) This can be also derived directly from Gersgorin's theorem, mentioned in the introduction.

As an illustration we give the following

Example 3. Each eigenvalue of the matrix A defined in Example 2 satisfies one of the inequalities:

$$\left|\frac{2}{3-\lambda}\right| \geqslant 1, \qquad \left|\frac{3}{5-\lambda}\right| + \left|\frac{1}{3-\lambda}\right| \geqslant 1.$$

THEOREM 2. *Let $A = (a_{ij})$, $i, j = 1, 2, \ldots$, be an infinite matrix with $a_{ii} \neq 0$ and suppose that there exists $p > 1$ such that*

$$(\mathrm{b}_5) \qquad \sum_{i=1}^{\infty}\Big(\sum_{j=1, j\neq i}^{\infty}\left|\frac{a_{ij}}{a_{ii}}\right|^p\Big)^{q/p} \leqslant 1.$$

Then $\lambda = 0$ is not an eigenvalue of A relative to $\tilde{l}^q$, where $1/p + 1/q = 1$.

Proof. Suppose to the contrary that there exists a point $x = (x_1, x_2, \ldots) \in \tilde{l}^q$ such that $x \neq 0$ and $Ax = 0$. Then

$$\sum_{j=1}^{\infty} a_{ij} x_j = 0$$

and thus

$$x_i = -\sum_{i=1, j\neq i}^{\infty}\frac{a_{ij}}{a_{ii}} x_j.$$

Hence, by Hölder's inequality

$$|x_i| \leqslant \Big(\sum_{j=1, j\neq i}^{\infty}\left|\frac{a_{ij}}{a_{ii}}\right|^p\Big)^{1/p}\Big(\sum_{j=1, j\neq i}^{\infty}|x_j|^q\Big)^{1/q}$$

and thus for $x \neq 0$ we get

$$\sum_{i=1}^{\infty}|x_i|^q < \sum_{i=1}^{\infty}\Big(\sum_{j=1, j\neq i}^{\infty}\left|\frac{a_{ij}}{a_{ii}}\right|^p\Big)^{q/p}\sum_{j=1}^{\infty}|x_j|^q,$$

which contradicts (b_5).

Remark 3. Let us note that in the case of $p = 2$ one obtains from Theorem 2 that

(b_6) if $A = (a_{ij})$, $i, j = 1, 2, \ldots$, is an infinite matrix with $a_{ii} \neq 0$ such that

$$\sum_{i=1}^{\infty}\sum_{j=1, j\neq i}^{\infty}\left|\frac{a_{ij}}{a_{ii}}\right|^2 \leqslant 1,$$

then $\lambda = 0$ is not eigenvalue of A relative to $\tilde{l}^2$.

As a consequence of (b_6) we get that each eigenvalue of an arbitrary infinite matrix $A = (a_{ij})$, $i, j = 1, 2, \ldots$, relative to the space $\tilde{l}^2$ satisfies the inequality

$$\sum_{i=1}^{\infty} \sum_{j=1, j \neq i}^{\infty} \left| \frac{a_{ij}}{a_{ii} - \lambda} \right|^2 > 1.$$

2. In this section we prove some theorems on infinite matrices related to Gersgorin's theorem quoted above. It will be shown also that for Banach spaces X and Y with a Schauder basis, condition (b_1) does imply that $A: X \to Y$ is one-to-one. First let us introduce the following

Definition. We say that a space Z whose points $z = (z_1, z_2, \ldots)$ are sequences of complex numbers has property (l) if and only if for each $z = (z_1, z_2, \ldots) \in Z$ there exists an index $i_0 = i_0(z)$ such that

$$|z_{i_0}| = \max_i |z_i|$$

(i.e. for each $z = (z_1, z_2, \ldots)$ there exists a coordinate z_{i_0} with $|z_{i_0}| \geqslant |z_i|$ for all $i = 1, 2, \ldots$). For instance each $\tilde{l}^p$ space has property (l). The space m does not have property (l).

THEOREM 3. *Let $A = (a_{ij})$ be an arbitrary matrix (finite or not) such that (b_1) holds, and let X have property (l). Then 0 is not an eigenvalue of A relative to X.*

Proof [3]. Suppose to the contrary that there exists a point $x = (x_1, x_2, \ldots) \neq 0$ of X such that

$$\sum_{j=1}^{\infty} a_{ij} x_j = 0 \quad \text{for all} \quad i = 1, 2, \ldots \tag{c}$$

Since $x \in X$ and X has property (l) there exists an index i_0 such that $|x_{i_0}| = \max |x_i|$. Then, by (b_1), we have

$$|a_{i_0 i_0}|\,|x_{i_0}| > \sum_{j \neq i_0} |a_{i_0 j}|\,|x_{i_0}| \geqslant \sum_{j \neq i_0} |a_{i_0 j}|\,|x_j|.$$

On the other hand, by (c) we have

$$|a_{i_0 i_0}|\,|x_{i_0}| \leqslant \sum_{j \neq i_0} |a_{i_0 j}|\,|x_j|,$$

which contradicts the foregoing inequality.

THEOREM 4. *Let $A: X \to Y$ be a linear continuous mapping of a (in general complex) Banach space X into a Banach space Y. Suppose that X has a Schauder basis consisting of unit vectors $e_1, e_2, \ldots$ and Y has Schauder basis consisting of unit vectors $\bar{e}_1, \bar{e}_2, \ldots$ and let us define for*

$$y = \sum_{i=1}^{\infty} y_j \bar{e}_j \in Y$$

[3] The idea of this proof is the same as that of Theorem 1 in [3].

the functional $f_j \in Y^*$ *by* $(f_j, y) = y_j$, ((f, y) *denotes the value of* f *at the point* $y \in Y$). *Write* $a_{ij} = (f_i, Ae_j)$ *and suppose that* (b_1) *holds for the matrix* (a_{ij}), $i, j = 1, 2, \ldots$

Then $A: X \to Y$ *is one-to-one.*

Proof. To the points

$$x = \sum_{i=1}^{\infty} x_i e_i \in X \quad \text{and} \quad y = \sum_{i=1}^{\infty} y_i \bar{e}_i \in Y$$

we make correspond the sequences $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ of their coordinates. Then X and Y may be considered as spaces of sequences of complex numbers (denoted also by X and Y).

The operator $A: X \to Y$ can be represented by the matrix $A = (a_{ij})$, $i, j = 1, 2, \ldots$, where $a_{ij} = (f_i, Ae_j)$, and the mapping $A: X \to Y$ is given by formula (a). Since X has a Schauder basis of unit vectors $e_1, e_2, \ldots$, we have for each

$$x = \sum_{i=1}^{\infty} x_i e_i = \lim_{n\to\infty} \sum_{i=1}^{n} x_i e_i$$

that $\left\{\sum_{i=1}^{n} x_i e_i\right\}$, $n = 1, 2, \ldots$, is a Cauchy sequence. Thus

$$|x_n| = ||x_n e_n|| = \left\|\sum_{i=1}^{n} x_i e_i - \sum_{i=1}^{n-1} x_i e_i\right\| \to 0 \quad \text{as} \quad n \to \infty.$$

It follows that X has property (l) and, by (b_1) and theorem 3, $A: X \to Y$ is one-to-one.

Example 4. Let

$$\sum_{m,n=-\infty}^{\infty} a_{mn} e^{2i\pi(ms+nt)}$$

be the Fourier series of a square integrable function $K(s, t)$ for $0 < s$, $t < 1$ and let us define $y = Ax$ for $x \in X = L^2\,[0, 1]$, by

$$y = y(s) = Ax = \int_0^1 K(s, t) x(t)\, dt.$$

Then $A: X \to X$ is a linear continuous mapping of X into itself. Putting

$$e_n = e_n(t) = e^{-2in\pi t}$$

and

$$(f_m, y) = \int_0^1 e^{-2im\pi s} y(s)\, ds, \qquad n, m = 0, \pm 1, \pm 2, \ldots,$$

we have $(f_m, Ae_n) = a_{mn}$ [4]. Thus by Theorem 4, if

$$|a_{mm}| > \sum_{\substack{n=-\infty \\ n \neq m}}^{\infty} |a_{mn}|,$$

then $A: X \to X$ is one-to-one. Fixing m_0 one can replace this inequality by

$$|a_{m\,m+m_0}| > \sum_{\substack{n=-\infty \\ n \neq m+m_0}}^{\infty} |a_{mn}|, \qquad m = 0, \pm 1, \pm 2, \ldots$$

In the next following Theorems 5 and 6 we will show cases in which the "strong" inequality (b_1) can be replaced by

$$(c_1) \qquad |a_{ii}| \geqslant \sum_{j \neq i} |a_{ij}| \quad \text{for all } i = 1, 2, \ldots,$$

and nevertheless $A: X \to Y$ remains one-to-one.

THEOREM 5. *Let $A = (a_{ij})$ be an arbitrary infinite matrix such that for every $i = 1, 2, \ldots$ there exist infinitely many indices $j = j(i)$ such that $a_{i,j(i)} \neq 0$ and (c_1) holds.*

Let X be a space whose points are sequences, $x = (x_1, x_2, \ldots)$ of complex numbers such that

$$\lim_{n \to \infty} |x_n| = 0.$$

Then $\lambda = 0$ is not an eigenvalue of A relative to X.

Proof. As in the proof of Theorem 3, suppose that there exists a point $x = (x_1, x_2, \ldots) \neq 0$ of X such that $Ax = 0$. Since $|x_n| \to 0$ as $n \to \infty$ there exists an index i_0 such that $|x_{i_0}| = \max_i |x_i|$. Then we have by (c_1) the inequalities

$$|a_{i_0 i_0} x_{i_0}| \geqslant \sum_{j \neq i_0} |a_{i_0 j} x_{i_0}| \geqslant \sum_{j \neq i_0} |a_{i_0 j} x_j|.$$

Since $a_{i_0 j(i_0)} \neq 0$ for infinitely many indices $j(i_0)$ and $|x_n| \to 0$, there exists an index j_0 such that $a_{i_0 j_0} \neq 0$ and $|x_{j_0}| < |x_{i_0}|$. Thus

$$|a_{i_0 j_0} x_{j_0}| < |a_{i_0 j_0} x_{i_0}|$$

and therefore

$$|a_{i_0 i_0} x_{i_0}| > \sum_{j \neq i_0} |a_{i_0 j} x_j|.$$

On the other hand, by

$$a_{i_0 i_0} x_{i_0} = - \sum_{j \neq i_0} a_{i_0 j} x_j$$

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[4] One can obviously enumerate the double sequence $\{a_{mn}\}$ using only positive integers $m, n = 1, 2, 3, \ldots$ as indices.

we have

$$|a_{i_0 i_0} x_{i_0}| \leqslant \sum_{j \neq i_0} |a_{i_0 j} x_j|,$$

which contradicts the foregoing inequality.

Similarly as in the proof of Theorem 4, one can prove by using Theorem 5 the following

THEOREM 6. *If* $A: X \to Y$ *satisfies the assumptions of Theorem* 4 *with* (b_1) *replaced by* (c_1) *and if for every* $i = 1, 2, \ldots$ *there are infinitely many indices* $j = j(i)$ *such that* $a_{ij} = (f_i, Ae_j) \neq 0$, *then* $A: X \to Y$ *is one-to-one.*

3. In this section we show how a generalization of Gersgorin's theorem and various sufficient conditions for an operator $A: X \to Y$ to be one-to-one can be derived from the following well known Theorem (d) (see for example, [4], Theorem 8.7.4, p. 213):

(d) If $U: X \to Y$ is a continuous linear operator of a Banach space X into a Banach space Y such that U^{-1} exists and if $V: X \to Y$ is such that

$$||V - U|| < \frac{1}{||U^{-1}||},$$

then V^{-1} exists.

Let now $A: X \to Y$ be a mapping given by an infinite matrix $A = (a_{ij})$, $a_{ii} \neq 0$, according to formula (a), where X and Y are Banach spaces whose points are sequences of complex numbers, and suppose that $X = Y$ (as a set but the norms may be different). Then the existence of a point $x = (x_1, x_2, \ldots) \in X$ such that

$$\sum_{j=1}^{\infty} a_{ij} x_j = 0$$

and $x \neq 0$ implies that

$$\sum_{i=1}^{\infty} \frac{a_{ij}}{a_{ii}} x_j = 0, \qquad i = 1, 2, \ldots \text{ and } x = (x_1, x_2, \ldots) \neq 0.$$

This can be expressed by saying that the operator $I + B$, where $I: X \to Y$ is the identity operator on X and $B = (b_{ij})$ is a matrix with

$$b_{ij} = (1 - \delta_{ij}) \frac{a_{ij}}{a_{ii}} \qquad \delta_{ij} = \begin{cases} 0 & \text{if} \quad i \neq j, \\ 1 & \text{if} \quad i = j, \end{cases}$$

is *not* one-to-one. In order to show that the mapping $A: X \to Y$ is one--to-one it is enough to prove (by theorem (d)) that

$$||B|| < \frac{1}{||I^{-1}||}.$$

This idea and the meaning of I and B will be used in the next following Theorem 7 and one more example.

THEOREM 7. *Let* $X = Y = \tilde{m}$. *Let* $A\colon X \to X$ *be an operator given by a matrix* (a_{ij}), $i, j = 1, 2, \ldots$, *according to formula* (a), *such that there exists a constant* $\delta < 1$ *satisfying*

$$\sup_i \sum_{j \neq i} \left| \frac{a_{ij}}{a_{ii}} \right| < \delta. \tag{d$_1$}$$

Then $A\colon X \to X$ *is one-to-one.*

Proof. The proof follows from theorem (d) by noting that the norm of $I^{-1}\colon X \to X$ is equal to 1, and by (d_1) we have $\|B\| < 1$.

Remark 4. Let us note that a theorem similar to Theorem 7 can be proved exactly in the same manner for *finite* matrices $A = (a_{ij})$, $i, j = 1, \ldots, \nu$, mapping the ν-dimensional space $\tilde{m}_\nu$ of all sequences $x = (x_1, x_2, \ldots, x_\nu)$ of ν complex numbers x_i, $i = 1, 2, \ldots, \nu$, into itself, with condition (d_1) replaced by (b). It follows that (b) implies $\det A \neq 0$, which is Gersgorin's theorem.

Choosing different norms in the linear spaces of all sequences $x = (x_1, x_2, \ldots, x_\nu)$ one can get different conditions for a matrix $A = (a_{ij})$ to satisfy $\det A \neq 0$. This fact is illustrated in the following

Example 6. Suppose that the matrix $A = (a_{ij})$, $i, j = 1, \ldots, \nu$, satisfies the inequality

$$\sum_{i=1}^{\nu} \sum_{j \neq i} \left| \frac{a_{ij}}{a_{ii}} \right|^2 < 1. \tag{$\overline{d}_1$}$$

Then $\det A \neq 0$ [5].

Indeed, consider the space $\tilde{l}^2_\nu$ of all sequences $x = (x_1, x_2, \ldots, x_\nu)$ with norm

$$\|x\| = \Big(\sum_{i=1}^{\nu} |x_i|^2\Big)^{1/2}.$$

Then $\|I^{-1}\| = 1$ and by $(\overline{d}_1)$ and the fact that $\|B\|$ is not greater than $\big(\sum_{i,j=1}^{\nu} |b_{ij}|^2\big)^{1/2}$ we have $\|B\| < 1$. Thus $\det A \neq 0$. For instance the matrix

$$A = \begin{pmatrix} 15 & 10 & 6 \\ 5 & 20 & 5 \\ 3 & 3 & 10 \end{pmatrix}$$

[5] This follows also from (b_6).

satisfies $(\bar{\mathrm{d}}_1)$ but does not satisfy (b). For the eigenvalues λ of every $\nu \times \nu$ matrix $A = (a_{ij})$ we have also

$$\sum_{i=1}^{\nu} \sum_{j \neq i} \left| \frac{a_{ij}}{a_{ii} - \lambda} \right|^2 \geqslant 1.$$

For example for the matrix

$$A = \begin{pmatrix} 4 & 4 & 4 \\ 1 & 4 & 1 \\ 1 & 4 & 4 \end{pmatrix}$$

we obtain $|4-\lambda| \leqslant \sqrt{51}$, and this is a better estimate than the one obtained by applying Gersgorin's result which gives only $|4-\lambda| \leqslant 8$.

Let us finally note that by considering the spaces $\tilde{l}_\nu^q$ of all sequences $x = (x_1, x_2, \ldots, x_n)$ with norm

$$\|x\| = \Big(\sum_{i=1}^{\nu} |x_i|^q\Big)^{1/q} \qquad (q \geqslant 1)$$

and I and B as mappings of $\tilde{l}_\nu^q$ into $\tilde{l}_\nu = \tilde{l}_\nu^1$ one can again obtain that $(\tilde{\mathrm{b}}_3)$ implies $\det A \neq 0$.

4. In this section some applications of the previously obtained results in quantum mechanics are indicated. Namely, let the Hamiltonian H of some system be given by an infinite matrix $H = (h_{ij})$, $i, j = 0, 1, \ldots$, considered as an operator on some set of infinite sequences. The possible energy values of the system are the eigenvalues of H (usually relative to $\tilde{l}^2$) and the main problem of perturbation theory is to estimate these eigenvalues. (The exact values of the eigenvalues are known in very few cases.) This is done by writing H as an infinite series

$$(*) \qquad H = H^{(0)} + \varepsilon H^{(1)} + \varepsilon^2 H^{(2)} + \ldots,$$

where $H^{(0)}$ is a diagonal matrix and ε a small parameter. Then, taking the first two or three terms of this series, one can estimate the deviation of eigenvalues of H from those of $H^{(0)}$. The above procedure is rarely justified since, because of the unboundedness of $H^{(n)}$, the series in $(*)$ usually does not converge. It is, however, easily seen that the theorems proved in the previous sections can be applied to obtain *strictly* some estimates on the eigenvalues of H.

We will illustrate this last statement on two examples.

Example 7. Let p and x be the matrices of the momentum and position as defined in [2], p. 326. The Hamiltonian matrix of the harmonic oscilator is then

$$H^{(0)} = \tfrac{1}{2}(p^2 + x^2) = (h_{ij}),$$

where $h_{ii} = (i + \frac{1}{2})$ for $i = 0, 1, 2, \ldots$, and $h_{ij} = 0$ for $i \neq j$.

Let us now take a small perturbation of the form εx^2 of $H^{(0)}$, i.e., let us consider the Hamiltonian $H = \frac{1}{2}(p^2+x^2)+\varepsilon x^2$. One can find the eigenvalues of H directly. However, one will not look here for the exact values of the eigenvalues of H but instead find some estimates on these eigenvalues. We have namely

$$H = \frac{1}{2}\begin{bmatrix} 1 & 0 & -\varepsilon\sqrt{2} & 0 & 0 & \dots \\ 0 & 3 & 0 & -\varepsilon\sqrt{2\cdot 3} & 0 & \dots \\ -\varepsilon\sqrt{2} & 0 & 5 & 0 & -\varepsilon\sqrt{3\cdot 4} & \dots \\ 0 & -\varepsilon\sqrt{2\cdot 3} & 0 & 7 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}.$$

Thus, by Theorem 3 applied to $A = H-\lambda I$ (considered as an operator on a subset of $\tilde{l}^2$) we see that each eigenvalue λ of H lies in some interval I_n where

$$I_n = \left\{x;\ \left|n+\frac{1}{2}-x\right| \leqslant \frac{\varepsilon}{2}[\sqrt{(n-1)n}+\sqrt{(n+1)(n+2)}],\ n = 0,1,2,\dots\right\}.$$

This does not give much information on λ, because for $n = n(\varepsilon)$ large enough, the intervals I_n and I_{n+1} have a non-empty intersection. Nevertheless, for example, for $\varepsilon = \frac{1}{2}$ one can easily show that no eigenvalue λ lies in the interval $\frac{1}{2}+\frac{1}{4}\sqrt{2} < x < \frac{3}{2}-\frac{1}{4}\sqrt{6}$. By remark 2 applied to $A = H$ we obtain that each eigenvalue of H relative to $\tilde{l}$ satisfies also one of the inequalities

$$\left|\frac{\varepsilon\sqrt{(n+1)(n+2)}}{2n+5-2\lambda}\right| + \left|\frac{\varepsilon\sqrt{(n-1)n}}{2n-3-2\lambda}\right| \geqslant 1, \qquad n = 0,1,2,\dots$$

Example 8. The Hamiltonian $H = \frac{1}{2}(p^2+x^2)+\varepsilon x^4$ is obtained in the theory of the anharmonic oscilator, see [2], p. 387. Writing $H = (h_{mn})$, we have $h_{m,n} = 0$ for $m-n \neq 0, \pm 2, \pm 4$ and H is a symmetric matrix with

$$h_{nn} = \left(n+\frac{1}{2}\right)+3\frac{\varepsilon}{2}\left(n^2+n+\frac{1}{2}\right),$$

$$h_{n,n+2} = \frac{\varepsilon}{2}(2n+3)\sqrt{(n+1)(n+2)}.$$

$$h_{n,n+4} = \frac{\varepsilon}{4}\sqrt{(n+1)(n+2)(n+3)(n+4)},$$

where $m, n = 0,1,2,\dots$ Applying again Theorem 3 to $A = H-\lambda I$

we obtain that each eigenvalue λ of H relative to $\tilde{l}^2$ is contained in some interval

$$I_n = \{x;\ |x-h_{nn}| \leqslant h_{n-2,n}+h_{n-4,n}+h_{n,n+2}+h_{n,n+4}\}, \qquad n = 0, 1, 2, \ldots$$

Thus, assuming that every such interval contains an eigenvalue of H (this can be justified by physical arguments) we obtain some information on the eigenvalues of H.

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TECHNION, ISRAEL INSTITUTE OF TECHNOLOGY

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