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ABSTRACT COVARIANT DERIVATIVE

BY

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For any manifold M of class C_∞ let $\mathscr{R}_M$ be the ring of all infinitely derivable functions on M.

M is completely determined by $\mathscr{R}_M$. Many notions concerning M are defined in terms of $\mathscr{R}_M$, and even the definition of manifolds M of class C_∞ can be formulated in terms of $\mathscr{R}_M$. A big part of the theory of manifolds, in particular of the theory of covariant differentation on M, is indeed a theory of $\mathscr{R}_M$.

It is not difficult to observe that in many definitions and theorems concerning M only algebraic properties of $\mathscr{R}_M$ play an essential part, viz. the fact that $\mathscr{R}_M$ is an algebraic ring.

This point of view suggests a possibility of the following generalization: Forget that $\mathscr{R}_M$ is the ring of infinitely derivable functions. Take an arbitrary commutative algebraic ring $\mathscr{R}$ instead of $\mathscr{R}_M$. Translate fundamental notions from the theory of manifolds of class C_∞ into the language of the theory of algebraic rings $\mathscr{R}$, as many as possible. Generalize theorems from differential geometry of manifolds to obtain theorems from the theory of algebraic rings.

This paper contains such a generalization concerning covariant differentation on manifolds.

It is obvious how to generalize fundamental notions from the intrinsic geometry of manifolds. The set $\mathscr{V}$ of all differential operators in $\mathscr{R}$ is a natural substitute for the module $\mathscr{V}_M$ of all vector fields defined on M and tangent to M. Covariant derivative is a mapping from $\mathscr{V}$ into the set of all endomorphisms in $\mathscr{V}$, etc.

It is less obvious how to generalize the theory of submanifolds M of a manifold N but it is also possible to make such a generalization. Besides $\mathscr{R}$ and $\mathscr{V}$ we have to introduce another module $\mathscr{W}$ ($\mathscr{V} \subset \mathscr{W}$) which is an abstract substitute of the module $\mathscr{W}_{M,N}$ of all vector fields defined on M and tangent to N. The module $\mathscr{W}_{M,N}$ contains all the necessary informations concerning the mutual connection between M and N.

The general theory of algebraic commutative rings, obtained in this way, is a kind of algebra of differential geometry. However, this algebra has also models essentialy different from the classical ones in differential geometry.

§§ 2-13 contain the exposition of a fragment of the algebra of differential geometry. Besides definitions of fundamental notions a few theorems are quoted; they are generalizations of fundamental theorems from differential geometry concerning the torsion tensor, the curvature tensor etc. Very often the theorems are given without any proof because their proofs can be obtained by a simple calculation and they are, in fact, the same as in modern books on differential geometry.

§ 14 treats of set-theoretical models of the theory.

§ 1. Terminology and notation. We shall consider a fixed algebraic commutative ring $\mathscr{R}$. It is not supposed that $\mathscr{R}$ has a unit element. Sometimes we shall assume that $\mathscr{R}$ has the following property:

(*) for every $a \in \mathscr{R}$ there exists a unique element $b \in \mathscr{R}$ such that $a = b+b$.

The element b will be denoted by $\frac{1}{2}a$. If $\mathscr{R}$ has the property (*), then $\mathscr{R}$ is said to be *dyadic*.

By a *module* we shall always understand an abelian group $\mathscr{W}$ (written additively) with a multiplication of elements of $\mathscr{R}$ by elements of $\mathscr{W}$, such that

(a) if $a \in \mathscr{R}$ and $W \in \mathscr{W}$, then $aW \in \mathscr{W}$;

(b) $(a+b)W = aW+bW$ for $a, b \in \mathscr{R}$ and $W \in \mathscr{W}$;

(c) $a(U+W) = aU+aW$ for $a \in \mathscr{R}$ and $U, W \in \mathscr{W}$;

(d) $a(bW) = (ab)W$ for $a, b \in \mathscr{R}$ and $W \in \mathscr{W}$.

If $\mathscr{R}$ has a unit element 1, we require also that $1W = W$ for every $W \in \mathscr{W}$.

The ring $\mathscr{R}$ itself is the simplest example of a module.

A module $\mathscr{W}$ is said to be *proper* provided

(1) $$\text{if } aW = 0 \text{ for every } W \in \mathscr{W}, \text{ then } a = 0.$$

For instance, if $\mathscr{R}$ has the unit element, then $\mathscr{R}$ is a proper module.

Let $\mathscr{W}$ and $\mathscr{W}'$ be modules. A mapping $L\colon \mathscr{W} \to \mathscr{W}'$ is said to be *additive* if

$$L(U+W) = L(U)+L(W) \quad \text{for} \quad U, W \in \mathscr{W}.$$

L is said to be *homogeneous* if

$$L(aW) = aL(W) \quad \text{for} \quad a \in \mathscr{R} \text{ and } W \in \mathscr{W}.$$

L is said to be *linear* if it is additive and homogeneous.

The same terminology is adopted for mappings

(2) $$L\colon \mathscr{W}_1\times\ldots\times\mathscr{W}_n \to W_{n+1},$$

where $\mathscr{W}_1, \ldots, \mathscr{W}_{n-1}$ are modules. It is obvious what we mean by saying that $L(W_1, \ldots, W_n)$ is *additive* (or *homogenous*, or *linear*) *in a variable* W_i, $i = 1, \ldots, n$. L is said to be *multilinear* or, more precisely, to be *n-linear*, provided $L(W_1, \ldots, W_n)$ is linear in all the variables $W_1, \ldots, W_n$. The set of all multilinear mappings (2) will be denoted by

(3) $$\mathfrak{L}(\mathscr{W}_1, \ldots, \mathscr{W}_n; \mathscr{W}_{n-1}).$$

Obviously, (3) is also a module with an obvious definition of addition and multiplication. If $\mathscr{W}$ is a module, then the module $\mathfrak{L}(\mathscr{W}; \mathscr{R})$ of all linear mappings from $\mathscr{W}$ into $\mathscr{R}$ will be denoted by $\mathscr{W}^*$. If $U\in\mathscr{W}^*$ and $W\in\mathscr{W}$, we shall sometimes write UW or WU instead of $U(W)$.

Let $\mathbf{T}$ be the smallest set such that

(e) the numbers 0 and 1 are in $\mathbf{T}$;

(f) if $\tau_1, \ldots, \tau_{n+1}$ are in $\mathbf{T}$, $n \geqslant 1$, then the sequence $\tau = (\tau_1, \ldots, \tau_{n+1})$ is in $\mathbf{T}$.

Elements in $\mathbf{T}$ are called *types*.

Let $\mathscr{W}$ be a module. For any type τ in $\mathbf{T}$ we define the *module τ-associated with $\mathscr{W}$* by induction as follows. By the module 0-associated with $\mathscr{W}$ we mean the module $\mathscr{R}$. By the module 1-associated with $\mathscr{W}$ we mean the module $\mathscr{W}$. If $\tau = (\tau_1, \ldots, \tau_{n+1})\in\mathbf{T}$, then by the module τ-associated with $\mathscr{W}$ we mean the module (3), where $\mathscr{W}_i$ is τ_i-associated with $\mathscr{W}$, $i = 1, \ldots, n+1$.

A module is said to be *associated with $\mathscr{W}$* if it is τ-associated with $\mathscr{W}$ for a $\tau\in\mathbf{T}$. It may happen that the same module is τ_1-associated and τ_2-associated with $\mathscr{W}$ for $\tau_1 \neq \tau_2$. However, we are interested only in the case where the mapping which assigns, to every $\tau\in\mathbf{T}$, the module τ-associated with $\mathscr{W}$ is one-to-one.

Multilinear mappings (2) will be often called *tensors*. If all the modules $\mathscr{W}_1, \ldots, \mathscr{W}_{n+1}$ are associated with a module $\mathscr{W}$, then every multilinear mapping (2) is called a *$\mathscr{W}$-tensor*.

A tensor $L\in\mathfrak{L}(\mathscr{W}, \mathscr{W}; \mathscr{W}')$ is said to be *symmetric* if $L(W_1, W_2) = L(W_2, W_1)$ for any $W_1, W_2\in\mathscr{W}$. It is said to be *skew-symmetric* if $L(W_1, W_2) = -L(W_2, W_1)$ for any $W_1, W_2\in\mathscr{W}$.

A tensor $G\in\mathfrak{L}(\mathscr{W}, \mathscr{W}; \mathscr{R})$ is said to be a *scalar product* on a module $\mathscr{W}$ if G is symetric and, for every $Z\in\mathscr{W}^*$, there exists exactly one $Y\in\mathscr{W}$ such that $G(X, Y) = Z(X)$ for every $X\in\mathscr{W}$.

If $G\in\mathfrak{L}(\mathscr{W}, \mathscr{W}; \mathscr{R})$, then for every fixed $Y\in\mathscr{W}$ the equation

$$Z(X) = G(X, Y)$$

defines a $Z \in \mathscr{W}^*$ and the equation

$$\tilde{G}(Y) = Z \tag{4}$$

defines a linear mapping $\tilde{G}: \mathscr{W} \to \mathscr{W}^*$. Thus a symmetric tensor G: $\mathscr{W} \times \mathscr{W} \to \mathscr{R}$ is a scalar product if and only if $\tilde{G}$ is a one-to-one mapping from $\mathscr{W}$ onto $\mathscr{W}^*$.

By a *basis* of a module $\mathscr{W}$ we shall understand any finite sequence $W_1, \ldots, W_n \in \mathscr{W}$ such that, for every $W \in \mathscr{W}$, there exists exactly one sequence $a^1, \ldots, a^n \in \mathscr{R}$ such that

$$W = a^1 W_1 + \ldots + a^n W_n,$$

or shortly: $W = a^i W_i$, using the known summation convention. The elements a^i are called the *coordinates* of $W \in \mathscr{W}$.

It is easy to check that if $\mathscr{W}$ has a basis, then $\mathscr{R}$ has a unit element 1. Moreover, there exists exactly one basis $W^1, \ldots, W^n$ in $\mathscr{W}^*$ such that

$$W^j(W_i) = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

This basis is said to be *dual* to the basis $W_1, \ldots, W_n$ in $\mathscr{W}$.

For any $W \in \mathscr{W}^*$ there exists exactly one sequence $a_1, \ldots, a_n \in \mathscr{R}$ such that

$$W = a_1 W^1 + \ldots + a_n W^n,$$

or briefly: $W = a_j W^j$. The elements a_j are said to be the *coordinates* of $W \in \mathscr{W}^*$.

For any $\mathscr{W}$-tensor L: $\mathscr{W}^r \times (\mathscr{W}^*)^s \to \mathscr{R}$ the elements

$$L_{i_1, \ldots, i_r}^{j_1, \ldots, j_s} = L(W_{i_1}, \ldots, W_{i_r}, W^{j_1}, \ldots, W^{j_s}).$$

are called *coordinates* of L.

Observe that if $\mathscr{W}$ has a basis, then $\mathscr{W}^{**}$ can be identified with $\mathscr{W}$.

By a *Lie algebra* we shall mean an abelian group $\mathscr{V}$ (written additively) with a multiplication $[X, Y]$ such that $[X, Y] \in V$ for all $X, Y \in V$ and

(g) $[X, Y] = 0$;

(h) $[Y, X] = -[X, Y]$ (skew symmetry);

(i) $[X + Y, Z] = [X, Z] + [Y, Z]$ (distributivity);

(j) $\big[X, [Y, Z]\big] + \text{cycl} = 0$ (Jacobi identity).

"cycl" denotes that, in the expression before this word, the letters X, Y, Z should be cyclically permuted and the three expressions so obtained should be added. Thus (j) is an abbreviation for

$$\big[X, [Y, Z]\big] + \big[Y, [Z, X]\big] + \big[Z, [X, Y]\big] = 0.$$

It follows from (h) and (i) that

$$[X, 0] = 0 = [0, X], \quad [-X, Y] = -[X, Y] = [X, -Y],$$
$$[X, Y+Z] = [X, Y]+[X, Z].$$

Let $\mathscr{V}$ and $\mathscr{V}'$ be Lie algebras. A mapping $L: \mathscr{V} \to \mathscr{V}'$ is a *Lie homomorphism* provided that

$$L(X+Y) = L(X)+L(Y) \quad \text{and} \quad L([X, Y]) = [L(X), L(Y)].$$

It is obvious what we mean by a *submodule* of a module and a *Lie subalgebra* of a Lie algebra: they are subgroups closed with respect to the multiplication under consideration.

§ 2. $\mathscr{R}$-vectors. By an *$\mathscr{R}$-vector* (or, simply, a *vector*) we shall mean any additive mapping $\partial: \mathscr{R} \to \mathscr{R}$ such that

(1) $$\partial(ab) = \partial a \cdot b + a \cdot \partial b \quad \text{for} \quad a, b \in \mathscr{R}.$$

The set of all *$\mathscr{R}$-vectors* will be denoted by $\mathfrak{V}(\mathscr{R})$. It is not void. In fact, the mapping from $\mathscr{R}$ onto the zero element of $\mathscr{R}$ is an $\mathscr{R}$-vector. It will be denoted by 0 and called the *zero vector*.

2.1. *$\mathfrak{V}(\mathscr{R})$ is a module with respect to the operations*

(2) $$(\partial_1+\partial_2)a = \partial_1 a+\partial_2 a,$$

(3) $$(a\partial)b = a\cdot\partial b.$$

The zero vector is the zero element of the module $\mathfrak{V}(\mathscr{R})$.

2.2. *$\mathfrak{V}(\mathscr{R})$ is a Lie algebra with respect to the addition* (2) *and the Lie multiplication*

(4) $$[\partial_1, \partial_2] = \partial_1\partial_2 - \partial_2\partial_1,$$

where $\partial_1\partial_2$ is the composite of $\partial_1, \partial_2 \in \mathfrak{V}(\mathscr{R})$, and similarly for $\partial_2\partial_1$. Moreover, for every $a \in \mathscr{R}$,

(5) $$[a\partial_1, \partial_2] = a[\partial_1, \partial_2] - \partial_2 a\cdot\partial_1,$$

(6) $$[\partial_1, a\partial_2] = a[\partial_1, \partial_2] + \partial_1 a\cdot\partial_2.$$

Observe that it follows directly from (1) that if $\mathscr{R}$ has a unit element 1, then

(7) $$\partial 1 = 0.$$

§ 3. Directional derivatives. Let ∂ be a vector and $\mathscr{W}$ a module. A mapping $\nabla: \mathscr{W} \to \mathscr{W}$ is said to be a *derivative on $\mathscr{W}$ in the direction ∂* if ∇ is additive and

(1) $$\nabla(aW) = \partial a\cdot W + a\cdot\nabla W \quad \text{for} \quad a \in \mathscr{R} \text{ and } W \in \mathscr{W}.$$

A mapping $\nabla\colon \mathscr{W} \to \mathscr{W}$ is said to be a *directional derivative* on $\mathscr{W}$ if it is a derivative on $\mathscr{W}$ in a direction ∂. The set of all directional derivatives on $\mathscr{W}$ will be denoted by $\mathfrak{D}(\mathscr{W})$. It is not void. For instance, the mapping from $\mathscr{W}$ onto the zero element of $\mathscr{W}$ is a derivative in the direction of the zero vector; this directional derivative will be denoted by 0 and called the *zero derivative*.

3.1. *$\mathfrak{D}(\mathscr{W})$ is a module with respect to the operations*

$$(\nabla_1+\nabla_2)W = \nabla_1 W+\nabla_2 W, \tag{2}$$

$$(a\nabla)W = a\cdot\nabla W, \tag{3}$$

and the zero derivative is the zero element of the module $\mathfrak{D}(\mathscr{W})$.

Moreover, if ∇_1 and ∇_2 are derivatives in directions ∂_1, ∂_2 respectively, then $\nabla_1+\nabla_2$ is a derivative in the direction $\partial_1+\partial_2$. If ∇ is a derivative in a direction ∂, then $a\nabla$ is a derivative in the direction $a\partial$.

3.2. *$\mathfrak{D}(\mathscr{W})$ is a Lie algebra with respect to the operation*

$$[\nabla_1, \nabla_2] = \nabla_1\nabla_2-\nabla_2\nabla_1, \tag{4}$$

where $\nabla_1\nabla_2$ is the composite of the mappings ∇_1, ∇_2, and similarly for $\nabla_2\nabla_1$. If ∇_1, ∇_2 are derivatives in directions ∂_1, ∂_2 respectively, then $[\nabla_1, \nabla_2]$ is a derivative in the direction $[\partial_1, \partial_2]$. Moreover

$$[a\nabla_1, \nabla_2] = a[\nabla_1, \nabla_2]-\partial_2 a\cdot\nabla_1, \tag{5}$$

$$[\nabla_1, a\nabla_2] = a[\nabla_1, \nabla_2]+\partial_1 a\cdot\nabla_2. \tag{6}$$

3.3. *If ∇ is a derivative on $\mathscr{W}$ in a direction ∂ and $L\in\mathfrak{L}(\mathscr{W};\mathscr{W})$, then $\nabla_2 = \nabla_1+L$ is also a derivative in the direction ∂. Conversely, if ∇_2 and ∇_1 are derivatives on $\mathscr{W}$ in the same direction ∂, then there exists an $L\in\mathfrak{L}(\mathscr{W};\mathscr{W})$ such that $\nabla_2 = \nabla_1+L$.*

3.4. *In order that an additive mapping from $\mathscr{W}$ into $\mathscr{W}$ be a derivative in the direction 0 it is necessary and sufficient that it be homogenous, i.e. linear.*

As we stated in § 1, the ring $\mathscr{R}$ itself is a module. It follows directly from § 2, (1), that every $\mathscr{R}$-vector ∂ is a directional derivative on $\mathscr{R}$, viz. a directional derivative in the direction ∂. Thus $\mathfrak{V}(\mathscr{R})$ is a subset of $\mathfrak{D}(\mathscr{R})$ (it easily follows from 3.3 or 3.4 that $\mathfrak{V}(\mathscr{R})$ is a proper subset of $\mathfrak{D}(\mathscr{R})$, except some trival cases).

3.5. *$\mathfrak{V}(\mathscr{R})$ is a submodule of the module $\mathfrak{D}(\mathscr{R})$ and a Lie subalgebra of the Lie algebra $\mathfrak{D}(\mathscr{R})$.*

§ 4. The case where $\mathscr{W}$ is proper. Let ∇ be a derivative on a module $\mathscr{W}$ in a direction ∂. It follows from § 3 (1) that the element $\partial a\cdot W$ is uniquely determined by ∇, a and W. If the module $\mathscr{W}$ is proper, the element ∂a is uniquely determined by ∇ and a. In other words, if W is proper, ∂ is uniquely determined by ∇. The mapping, which assigns, to every $\nabla \in \mathfrak{D}(\mathscr{W})$, the unique $\partial \in \mathfrak{V}(\mathscr{R})$ such that ∇ is a derivative in the direction ∂, will be called the *canonical mapping*. It easily follows from 3.3 that the canonical mapping is never one-to-one, except some trival cases.

4.1. *If $\mathscr{W}$ is a proper module, the canonical mapping from $\mathfrak{D}(\mathscr{W})$ into $\mathfrak{V}(\mathscr{R})$ is linear and is a Lie homomorphism.*

§ 5. Natural extensions. A directional derivative ∇ in $\mathscr{W}$ determines, in a natural way, a directional derivative on every module associated with $\mathscr{W}$. More precisely,

5.1. *Let ∇ be a derivative on a module $\mathscr{W}$ in a direction ∂. For every type $\tau \in \mathbf{T}$ there exists a unique derivative ∇_τ in the direction ∂ on the module τ-associated with $\mathscr{W}$, such that*

(a) *if $\tau = 0$, then $\nabla_\tau = \partial$;*

(b) *if $\tau = 1$, then $\nabla_\tau = \nabla$;*

(c) *if $\tau = (\tau_1, \ldots, \tau_{n+1})$, and $L \in \mathfrak{L}(\mathscr{W}_1, \ldots, \mathscr{W}_n; \mathscr{W}_{n+1})$, where $\mathscr{W}_i$ is the module τ_i-associated with $\mathscr{W}$ $(i = 1, \ldots, n+1)$, then*

$$(1) \qquad (\nabla_\tau L)(W_1, \ldots, W_n) = \nabla_{\tau_{n+1}}\big(L(W_1, \ldots, W_n)\big) - \\ -\big(L(\nabla_{\tau_1} W_1, W_2, \ldots, W_n) + \ldots + L(W_1, \ldots, W_{n-1}, \nabla_{\tau_n} W_n)\big)$$

for all $W_1 \in \mathscr{W}_1, \ldots, W_n \in \mathscr{W}_n$.

Clearly conditions (a)-(c) uniquely determine a mapping ∇_τ from the module τ-associated with $\mathscr{W}$ into itself. It suffices to verify that if $\tau = (\tau_1, \ldots, \tau_{n-1}) \in \mathbf{T}$ and $\nabla_{\tau_1}, \ldots, \nabla_{\tau_n}$ are derivatives in the direction ∂, so is ∇_τ. The easy proof is left to the reader.

The derivative ∇_τ is called the *natural τ-extension* (or simply, the *natural extension*) of ∇ on the module τ-associated with $\mathscr{W}$.

As we stated in § 1, we are interested only in the case where the mapping which assigns, to every type τ, the module τ-associated with W is one-to-one. In this case we can write ∇ instead of ∇_τ without any misunderstanding. After this convention

$$(2) \qquad \nabla a = \partial a \text{ for } a \in \mathscr{R}.$$

Formula (1) for the directional derivative of a $\mathscr{W}$-tensor L can be written in the form

$$(3) \qquad \nabla\big(L(W_1, \ldots, W_n)\big) = (\nabla L)(W_1, \ldots, W_n) + \\ + L(\nabla W_1, W_2, \ldots, W_n) + \ldots + L(W_1, \ldots, W_{n-1}, \nabla W_n).$$

5.2. *Let $\mathscr{W}'$ be a module associated with $\mathscr{W}$. If $U \in \mathfrak{L}(\mathscr{W}; \mathscr{W}')$, then*

$$\nabla\big(U(W)\big) = (\nabla U)W + U(\nabla W)) \quad \textit{for} \quad W \in \mathscr{W}.$$

If $L(U, W) = U(W)$ *for every* $U \in \mathfrak{L}(\mathscr{W}; \mathscr{W}')$ *and every* $W \in \mathscr{W}$, *then* $\nabla L = 0$.

5.3. *For any* $\mathscr{W}$*-tensor* $L \in \mathfrak{L}\big(\mathscr{W}_1, \ldots, \mathscr{W}_n; \mathfrak{L}(\mathscr{W}_{n+1}, \mathscr{W}_{n+2})\big)$, *let*

$$\bar{L}(W_1, \ldots, W_n, W_{n+1}) = L(W_1, \ldots, W_n)W_{n+1} \qquad \text{for} \quad W_1 \in \mathscr{W}_1, \ldots, W_{n+1} \in \mathscr{W}_{n+1}.$$

Then $\bar{L} \in \mathfrak{L}(\mathscr{W}_1, \ldots, \mathscr{W}_{n+1}; \mathscr{W}_{n+2})$ *and*

$$(\nabla\bar{L})(W_1, \ldots, W_{n+1}) = (\nabla L)(W_1, \ldots, W_n)W_{n+1}.$$

Theorem 5.3 says that the two operations, differentiation and adding the variable W_{n+1} to L, are commutative.

5.4. *For any* $\mathscr{W}$*-tensor* $L \in \mathfrak{L}(\mathscr{W}_1, \ldots, \mathscr{W}_n; \mathscr{W}_{n+1})$, *let*

$$L^*(W_1, \ldots, W_{n-1}) = L(W_1, \ldots, W_n) \quad \text{for all} \quad W_1 \in \mathscr{W}_1, \ldots, W_{n-1} \in \mathscr{W}_{n-1}$$

and for a fixed element $W_n \in \mathscr{W}_n$. *Then* $L^* \in \mathfrak{L}(\mathscr{W}_1, \ldots, \mathscr{W}_{n-1}; \mathscr{W}_{n+1})$ *and*

$$(\nabla L^*)(W_1, \ldots, W_{n-1}) = (\nabla L)(W_1, \ldots, W_{n-1}, W_n) + L(W_1, \ldots, W_{n-1}, \nabla W_n).$$

Theorem 5.4 says that, in general, ∇L^* is not equal to ∇L, i.e. the two operations, differentiation and fixation of a variable, are not commutative.

If $G \in \mathfrak{L}(\mathscr{W}, \mathscr{W}; \mathscr{R})$ is symmetric and $L \in \mathfrak{L}(\mathscr{W}; \mathscr{W})$, then GL will denote a $\mathscr{W}$-tensor $GL \in \mathfrak{L}(\mathscr{W}, \mathscr{W}; \mathscr{R})$ defined as follows:

$$GL(U, W) = G(LU, W) \quad \text{for} \quad U, W \in \mathscr{W}. \tag{4}$$

Observe that for any $G \in \mathfrak{L}(\mathscr{W}, \mathscr{W}; \mathscr{R})$

$$(\nabla G)(U, W) = \partial\big(G(U, W)\big) - G(\nabla U, W) - G(U, \nabla W). \tag{5}$$

Thus, if G is symmetric, so is ∇G.

5.5. *If* $G \in \mathfrak{L}(\mathscr{W}, \mathscr{W}; \mathscr{R})$ *is symmetric and* $L \in \mathfrak{L}(\mathscr{W}; \mathscr{W})$, *then*

$$\nabla(GL) = (\nabla G)L + G\nabla L.$$

In fact, by (3),

$$\begin{aligned}
\big(\nabla(GL)\big)(U, W) &= \partial\big(GL(U, W)\big) - GL(\nabla U, W) - GL(U, \nabla W) \\
&= \partial\big(G(LU, W)\big) - G(L\nabla U, W) - G(LU, \nabla W) \\
&= (\nabla G)(LU, V) + G\big(\nabla(LU), W\big) + G(LU, \nabla W) - \\
&\quad - G(L\nabla U, W) - G(LU, \nabla W) \\
&= (\nabla G)(LU, W) + G\big((\nabla L)U, W\big).
\end{aligned}$$

5.6. *Let* ∇, ∇_1, ∇_2 *be derivatives on* $\mathscr{W}$ *in directions* ∂, ∂_1, ∂_2 *respectively, let* $a \in \mathscr{R}$ *and let* $\mathscr{V}$ *be a module* τ*-associated with* $\mathscr{W}$.

$a\nabla_\tau$ is the natural extension of the derivative $a\nabla$ (in the direction $a\partial$) over $\mathscr{V}$.

$(\nabla_1)_\tau + (\nabla_2)_\tau$ is the natural extension of the derivative $\nabla_1 + \nabla_2$ (in the direction $\partial_1 + \partial_2$) over $\mathscr{V}$.

$[(\nabla)_\tau, (\nabla_2)_\tau]$ is the natural extension of the derivative $[\nabla_1, \nabla_2]$ (in the direction $[\partial_1, \partial_2]$) over $\mathscr{V}$.

If $\mathscr{W}$ is a proper module, then ∂ is uniquely determined by the directional derivative ∇ on $\mathscr{W}$ (see § 4). Consequently, by 5.1, ∇_τ is uniquely determined by ∇ itself, for every module $\mathscr{V}$ τ-associated with $\mathscr{W}$. It follows directly from 5.5 that

5.7. *If $\mathscr{W}$ is a proper module, and $\mathscr{W}'$ is a module associated with $\mathscr{W}$, then the mapping which assigns $\nabla_\tau \in \mathfrak{D}(\mathscr{W}')$ to every $\nabla \in \mathfrak{D}(\mathscr{W})$ is a linear Lie homomorphism from $\mathfrak{D}(\mathscr{W})$ into $\mathfrak{D}(\mathscr{W}')$.*

§ 6. Covariant derivative. Let $\mathscr{W}$ be a module. By a *covariant derivative* in $\mathscr{W}$ we mean any mapping

$$\nabla : \mathscr{V} \to \mathfrak{D}(\mathscr{W}) \tag{1}$$

(the value of ∇ at $X \in \mathscr{V}$ will be denoted by ∇_X) such that

(a) $\mathscr{V}$ is a submodule of $\mathfrak{V}(\mathscr{R})$ and a Lie subalgebra of $\mathfrak{V}(\mathscr{R})$;

(b) ∇ is linear;

(c) for every $X \in \mathscr{V}$, ∇_X is a derivative on $\mathscr{W}$ in the direction X.

If $\mathscr{W}$ is a proper module and Δ denotes the canonical mapping from $\mathfrak{D}(\mathscr{W})$ into $\mathscr{V}(\mathscr{R})$, then condition (c) can be formulated as follows:

$$\Delta \nabla = I,$$

where I is the identity mapping on $\mathscr{V}$.

Since this moment $\mathscr{V}$ will always denote a fixed subset of $\mathfrak{V}(\mathscr{R})$ satisfying (a) (the case where $\mathscr{V} = \mathfrak{V}(\mathscr{R})$ is, of course, admitted), and ∇ will always denote a fixed covariant derivative in a module $\mathscr{W}$, i.e. a mapping (1) satisfying (b) and (c). The letters X, Y, Z, V will denote elements of $\mathscr{V}$. The letters U, W will denote elements of $\mathscr{W}$, unless the contrary is explicitely stated.

According to the convention from § 5, if $\mathscr{W}'$ is the module τ-associated with $\mathscr{W}$, then $(\nabla_X)_\tau$ will denote the natural extension of the directional derivative ∇_X over the module $\mathscr{W}'$. Often the natural extension $(\nabla_X)_\tau$ will be denoted by the symbol $\nabla_{X|\mathscr{W}'}$ or by ∇_X for short. Thus the fundamental identity § 5, (1) or (3), for differentiation of $\mathscr{W}$-tensors can be now written as

$$\nabla_X\big(L(W_1, \ldots, W_n)\big) = (\nabla_X L)(W_1, \ldots, W_n) + \\ + L(\nabla_X W_1, \ldots, W_n) + \ldots + L(W_1, \ldots, \nabla_X W_n) \tag{2}$$

for any $L \in \mathfrak{L}(\mathscr{W}_1, \ldots, \mathscr{W}_n; \mathscr{W}_{n+1})$, $X \in \mathscr{V}$ and $W_1 \in \mathscr{W}_1, \ldots, W_n \in \mathscr{W}_n$, where $\mathscr{W}_1, \ldots, \mathscr{W}_{n+1}$ are modules associated with $\mathscr{W}$.

If $\mathscr{W}_1, \ldots, \mathscr{W}_{n+1}$ are modules associated with $\mathscr{W}$ and $L \in \mathfrak{L}(\mathscr{W}_1, \ldots, \mathscr{W}_n; \mathscr{W}_{n+1})$, then ∇L will now denote the mapping

(3) $$(\nabla L)(X, W_1, \ldots, W_n) = (\nabla_X L)(W_1, \ldots, W_n)$$

for $X \in \mathscr{V}$ and $W_1 \in \mathscr{W}_1, \ldots, W_n \in \mathscr{W}_n$.

It follows from (b) that, under the above hypothesis,

6.1. *The mapping $\nabla L: \mathscr{V} \times \mathscr{W}_1 \times \ldots \times \mathscr{W}_n \to \mathscr{W}_{n+1}$ is multilinear, that is*

$$\nabla L \in \mathfrak{L}(\mathscr{V}, \mathscr{W}_1, \ldots, \mathscr{W}_n; \mathscr{W}_{n+1}).$$

∇L will be called the *covariant derivative of the $\mathscr{W}$-tensor L.*

It follows from (c) and 5.1 (a) that

$$\nabla_{X|\mathscr{R}} = X \quad \text{for every } X \in \mathscr{V}.$$

According to the general convention adopted in § 5, p. 257, we shall often write $\nabla_X a$ instead of $X(a)$ (or Xa) for every $X \in \mathscr{V}$ and $a \in \mathscr{R}$. Following the notation from § 2-§ 5, we shall also write $\partial_X a$ instead of $X(a)$ for $X \in \mathscr{V}$ and $a \in \mathscr{R}$. Thus

(4) $$\partial_X = X \quad \text{for every } X \in \mathscr{V}.$$

The fundamental equation § 3 (1) can be now written as

(5) $$\nabla_X(aW) = \partial_X a \cdot W + a \cdot \nabla_X W$$

for $a \in \mathscr{R}$, $X \in \mathscr{V}$ and $W \in \mathscr{W}$. Identity (5) holds also if W is an element of a module $\mathscr{W}'$ τ-associated with $\mathscr{W}$, and ∇_X stands for $(\nabla_X)_\tau$.

Notice that by the convention (4),

(6) $$[\partial_X, \partial_Y] = \partial_{[X,Y]}.$$

The following theorem is an analogue of 3.3:

6.2. *If $L \in \mathfrak{L}(\mathscr{V}; \mathfrak{L}(\mathscr{W}; \mathscr{W}))$, then $\bar{\nabla} = \nabla + L$ is also a covariant derivative in $\mathscr{W}$. Conversely, if $\bar{\nabla}: \mathscr{V} \to \mathfrak{D}(\mathscr{W})$ is a covariant derivative, then there exists an $L \in \mathfrak{L}(\mathscr{V}; \mathfrak{L}(\mathscr{W}; \mathscr{W}))$ such that $\bar{\nabla} = \nabla + L$.*

§ 7. The curvature tensor. The covariant derivative $\nabla: \mathscr{V} \to \mathfrak{D}(\mathscr{W})$ is not supposed to be a Lie homomorphism from the Lie algebra $\mathscr{V}$ (see § 6 (a)) into the Lie algebra $\mathfrak{D}(\mathscr{W})$ (see 3.2). Consequently, for given $X, Y \in \mathscr{V}$, the mapping

(1) $$R_{X,Y} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}: \mathscr{W} \to \mathscr{W}$$

is not, in general, the zero mapping. By definition

(2) $$R_{X,Y} U = \nabla_X \nabla_Y U - \nabla_Y \nabla_X U - \nabla_{[X,Y]} U \in \mathscr{W} \quad \text{for} \quad U \in \mathscr{W}.$$

7.1. *The mapping* $R_{X,Y}\colon \mathscr{W} \to \mathscr{W}$ *is linear, i.e.* $R_{X,Y} \in \mathfrak{L}(\mathscr{W};\mathscr{W})$ *for any* $X, Y \in \mathscr{V}$.

The letter R will denote the mapping which assigns $R_{X,Y}$ for any $X, Y \in \mathscr{V}$.

7.2. *The mapping* $R\colon \mathscr{V}\times\mathscr{V} \to \mathfrak{L}(\mathscr{W};\mathscr{W})$ *is bilinear, i.e.* R *is an element of* $\mathfrak{L}\big(\mathscr{V},\mathscr{V};\mathfrak{L}(\mathscr{W};\mathscr{W})\big)$. *It is skew symmetric, i.e.*

$$R_{Y,X} = -R_{X,Y}. \tag{3}$$

The mapping R will be called the *curvature tensor* of ∇.

For any fixed $Y, Z \in \mathscr{V}$, $R_{Y,Z}$ is an element of the module $\mathscr{W}' = \mathfrak{L}(\mathscr{W};\mathscr{W})$ which is (1,1)-associated with $\mathscr{W}$. Hence, for any $X \in \mathscr{V}$, we can form the directional derivative $\nabla_X R_{Y,Z}$ (more precisely, the directional derivative $(\nabla_X)_\tau R_{Y,Z}$, where $(\nabla_X)_\tau$ is the natural τ-extension of ∇_X over $\mathscr{W}' = \mathfrak{L}(\mathscr{W};\mathscr{W})$, $\tau = (1,1)$). By § 6 (2),

$$(\nabla_X R_{Y,Z})(U) = \nabla_X(R_{Y,Z}U) - R_{Y,Z}(\nabla_X U) \quad \text{for} \quad U \in \mathscr{W}, \tag{4}$$

that is,

$$\nabla_X R_{Y,Z} = \nabla_X\nabla_Y\nabla_Z - \nabla_X\nabla_Z\nabla_Y - \nabla_X\nabla_{[Y,Z]} - \nabla_Y\nabla_Z\nabla_X + \nabla_Z\nabla_Y\nabla_X + \nabla_{[Y,Z]}\nabla_X. \tag{5}$$

7.3. *For any* $X, Y, Z \in \mathscr{V}$

$$\nabla_X R_{Y,Z} + R_{X,[Y,Z]} + \mathrm{cycl} = 0.$$

We recall (see p. 254) that "cycl" denotes that in the expressions $\nabla_X R_{Y,Z}$, $R_{X,[Y,Z]}$ the letters should be cyclically permuted and the six expressions obtained in this way should be added.

The proof is based on the fact that any sum followed by "+ cycl" remains unchanged if we perform a cyclic permutation of X, Y, Z in a summand. Therefore

$$\begin{aligned}
&\nabla_X R_{Y,Z} + \mathrm{cycl}\\
&= \nabla_X\nabla_Y\nabla_Z - \nabla_X\nabla_Z\nabla_Y - \nabla_X\nabla_{[Y,Z]} - \nabla_Y\nabla_Z\nabla_X + \nabla_Z\nabla_Y\nabla_X + \nabla_{[Y,Z]}\nabla_X + \mathrm{cycl}\\
&= \nabla_X\nabla_Y\nabla_Z - \nabla_X\nabla_Z\nabla_Y - \nabla_X\nabla_Y\nabla_Z + \nabla_X\nabla_Z\nabla_Y - R_{X,[Y,Z]} - \nabla_{[X,[Y,Z]]} + \mathrm{cycl}\\
&= -R_{[X,[Y,Z]]} + \mathrm{cycl},
\end{aligned}$$

since $\nabla_{[X,[Y,Z]]} + \mathrm{cycl} = \nabla_{[X,[Y,Z]]+\mathrm{cycl}} = 0$ by §6 (b) and the Jacobi identity § 1 (j).

Observe now that, by the definition (1), $R_{X,Y}$ is a difference of two directional derivatives: the derivative $[\nabla_X, \nabla_Y]$ in the direction $[X, Y]$ (see 3.2) and the derivative $\nabla_{[X,Y]}$ in the direction $[X, Y]$. Consequently (see 3.1) $R_{X,Y}$ is also a directional derivative, viz. a derivative in the direction $0 \in \mathfrak{V}(\mathscr{R})$ (the last property follows also directly from 7.1 and 3.4). According to a general convention from § 5, the natural extension $R_{X,Y|\mathscr{W}'}$

of $R_{X,Y}$ over a module τ-associated with $\mathscr{W}$ will be denoted by the same letter $R_{X,Y}$. By 5.5 and 5.1

(6) $$R_{X,Y}L = \nabla_X\nabla_Y L - \nabla_Y\nabla_X L - \nabla_{[X,Y]}L,$$

(7) $$R_{X,Y}\big(L(W_1, \dots, W_n)\big) = (R_{X,Y}L)(W_1, \dots, W_n) + \\ + L(R_{X,Y}W_1, W_2, \dots, W_n) + \dots + L(W_1, \dots, W_{n-1}, R_{X,Y}W_n)$$

for any $L \in \mathfrak{L}(\mathscr{W}_1, \dots, \mathscr{W}_n; W_{n+1})$ and $X, Y \in \mathscr{V}$, $W_1 \in \mathscr{W}_1, \dots, W_n \in \mathscr{W}_n$ where $\mathscr{W}_1, \dots, \mathscr{W}_{n+1}$ are modules associated with $\mathscr{W}$.

Since $R_{X,Y}$ is a derivation in the direction 0, we have $R_{X,Y|\mathscr{R}} = 0$ by 5.1 (a). Consequently, assuming $\mathscr{W}_{n+1} = \mathscr{R}$ in (7) we get the identity

(7′) $$(R_{X,Y}L)(W_1, \dots, W_n) + L(R_{X,Y}W_1, W_2, \dots, W_n) + \\ + L(W_1, \dots, W_{n-1}, R_{X,Y}W_n) = 0$$

for any $L \in \mathfrak{L}(\mathscr{W}_1, \dots, \mathscr{W}_n; \mathscr{R})$ and $X, Y \in \mathscr{V}$, $W_1 \in \mathscr{W}_1, \dots, W_n \in \mathscr{W}_n$ where $\mathscr{W}_1, \dots, \mathscr{W}_n$ are modules associated with $\mathscr{W}$.

In particular, if $G \in L(\mathscr{W}, \mathscr{W}; \mathscr{R})$, then

(8) $$(R_{X,Y}G)(U, W) + G(R_{X,Y}U, W) + G(U, R_{X,Y}W) = 0 \quad \text{for} \quad X, Y \in \mathscr{V} \text{ and } U, W \in \mathscr{W}.$$

For any symmetric $G \in \mathfrak{L}(\mathscr{W}, \mathscr{W}; \mathscr{R})$ we shall denote by GR the mapping

(9) $$GR(X, Y, U, W) = G(R_{X,Y}U, W) = G(W, R_{X,Y}U) \quad \text{for} \quad X, Y \in \mathscr{V} \text{ and } U, W \in \mathscr{W}.$$

7.4. *For any symmetric* $G \in \mathfrak{L}(\mathscr{W}, \mathscr{W}; \mathscr{R})$,

(10) $$GR \in \mathfrak{L}(\mathscr{V}, \mathscr{V}, \mathscr{W}, \mathscr{W}; \mathscr{R})$$

and

(11) $$GR(Y, X, U, W) = -GR(X, Y, U, W).$$

If $R_{X,Y}G = 0$, *in particular if* $\nabla G = 0$, *then*

(12) $$GR(X, Y, W, U) = -GR(X, Y, U, W).$$

(12) follows direcly from (8). Observe that if $\nabla G = 0$ (i.e. $\nabla_X G = 0$ for every $X \in \mathscr{V}$), then $R_{X,Y}G = 0$ by (6).

According to § 5 (4), for any $X, Y \in \mathscr{V}$ the symbol $GR_{X,Y}$ will denote the $\mathscr{W}$-tensor

$$GR_{X,Y}(U, W) = G(R_{X,Y}U, W) = GR(X, Y, U, W) \quad \text{for} \quad U, W \in \mathscr{W}.$$

By definition, $GR_{X,Y} \in \mathfrak{L}(\mathscr{W}, \mathscr{W}; \mathscr{R})$.

7.5. *If* $G \in \mathfrak{L}(\mathscr{W}, \mathscr{W}; \mathscr{R})$ *is symmetric, then*

(13) $$GR_{Y,X} = -GR_{X,Y}.$$

If moreover $\nabla G = 0$, *then* $GR_{X,Y}$ *is skew symmetric,*

$$GR_{X,Y}(W, U) = -GR_{X,Y}(U, W) \quad \textit{for} \quad U, W \in \mathscr{W},$$

and

(14) $$\nabla_X GR_{Y,Z} + GR_{X,[Y,Z]} + \mathrm{cycl} = 0 \quad \textit{for} \quad X, Y, Z \in \mathscr{V}.$$

This follows directly from 7.2, 7.4, 7.3 and 5.5.

§ 8. The case $\mathscr{V} \subset \mathscr{W}$**.** For applications to differential geometry the most important case is when

(a) $\mathscr{V}$ is a submodule of $\mathscr{W}$,

or even the case where $\mathscr{V} = \mathscr{W}$. The case $\mathscr{V} = \mathscr{W}$ will be discussed later on (§ 9). Since this moment we assume that (a) holds.

By (a), the following mapping $T\colon \mathscr{V} \times \mathscr{V} \to \mathscr{W}$,

(1) $$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \quad \text{for} \quad X, Y \in \mathscr{V},$$

is well defined. T is said to be the *torsion tensor the of covariant derivative* ∇. The covariant derivative ∇ is said to be *symmetric* provided $T = 0$.

8.1. T *is bilinear, i.e.* $T \in \mathfrak{L}(\mathscr{V}, \mathscr{V}; \mathscr{W})$, *and skew-symmetric*:

(2) $$T(Y, X) = -T(Y, X).$$

8.2. *For any* $X, Y, Z \in \mathscr{V}$

(3) $$R_{X,Y}Z + \mathrm{cycl} = \nabla_X\big(T(Y, Z)\big) + T(X, [Y, Z]) + \mathrm{cycl}.$$

In particular, if ∇ *is symmetric, then*

(4) $$R_{X,Y}Z + \mathrm{cycl} = 0.$$

In fact,

$$\begin{aligned} R_{X,Y}Z + \mathrm{cycl} &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z + \mathrm{cycl} \\ &= \nabla_X(\nabla_Y Z - \nabla_Z Y) - \nabla_{[Y,Z]} X + \mathrm{cycl} \\ &= \nabla_X\big(T(Y, Z)\big) + \nabla_X [Y, Z] - \nabla_{[Y,Z]} X + \mathrm{cycl} \\ &= \nabla_X\big(T(Y, Z)\big) + T(X, [Y, Z]) + \nabla_{[X,[Y,Z]]} + \mathrm{cycl} \\ &= \nabla_X\big(T(Y, Z)\big) + T(X, [Y, Z]) + \mathrm{cycl} \end{aligned}$$

since $\nabla_{[X,[Y,Z]]} + \mathrm{cycl} = \nabla_{[X,[Y,Z]] + \mathrm{cycl}} = 0$ by § 6 (b) and § 1 (j).

8.3. *If $G \in \mathfrak{L}(\mathscr{W}, \mathscr{W}; \mathscr{R})$ is symmetric, then for any $X, Y, Z \in \mathscr{V}$*

$$(5)\quad \begin{aligned} 2G(\nabla_X Y, Z) = {} & \partial_X G(Y, Z) + \partial_Y G(Z, X) - \partial_Z G(X, Y) - \\ & - (\nabla_X G)(Y, Z) - (\nabla_Y G)(Z, X) + (\nabla_Z G)(X, Y) + \\ & + G([X, Y], Z) - G([Y, Z], X) + G([Z, X], Y) + \\ & + G\big(T(X, Y), Z\big) - G\big(T(Y, Z), X\big) + G\big(T(Z, X), Y\big). \end{aligned}$$

In particular, if ∇ is symmetric and $\nabla G = 0$, then

$$(6)\quad \begin{aligned} 2G(\nabla_X Y, Z) = {} & \partial_X G(Y, Z) + \partial_Y G(Z, X) - \partial_Z G(X, Y) + \\ & + G([X, Y], Z) - G([Y, Z], X) + G([Z, X], Y). \end{aligned}$$

To prove (5) it suffices to calculate

$$\partial_X G(Y, Z) + \partial_Y G(Z, X) - \partial_Z G(X, Y)$$

using § 5 (3), and identity (1).

8.4. *Let $G \in \mathfrak{L}(\mathscr{W}, \mathscr{W}; \mathscr{R})$ be symmetric, ∇ be symmetric and $\nabla G = 0$. Then*

$$(7)\quad GR(X, Y, Z, W) + GR(Y, Z, X, W) + GR(Z, X, Y, W) = 0$$

for $X, Y, Z \in \mathscr{V}$ *and* $W \in \mathscr{W}$,

and

$$(8)\quad GR(X, Y, Z, V) = GR(Z, V, X, Y) \quad \text{for} \quad X, Y, Z, V \in \mathscr{V}.$$

(7) follows from 8.2. (8) follows from (7) and 7.4 (11-12) by a known algebraic theorem on four-linear mappings.

8.5. *If $L \in \mathfrak{L}\big(\mathscr{V}; \mathfrak{L}(\mathscr{W}; \mathscr{W})\big)$, then*

$$\overline{T}(X, Y) = T(X, Y) + L(X)Y - L(Y)X \quad \text{for} \quad X, Y \in \mathscr{V}$$

is the torsion tensor of the covariant derivative $\overline{\nabla} = \nabla + L$.

An element $X \in \mathscr{V}$ is said to be *∇-geodesic* if $X \neq 0$ and $\nabla_X X = 0$.

§ 9. The case of $\mathscr{V} = \mathscr{W}$. In this section we assume that $\mathscr{V}$ is identical with $\mathscr{W}$.

In that case the curvature tensor R is a $\mathscr{V}$-tensor and therefore we can form the directional derivative $\nabla_X R$ (more precisely, the directional derivative $(\nabla_X)_\tau R$ where $(\nabla_X)_\tau$ is the natural extension of ∇_X over the module $\mathscr{W}' = \mathfrak{L}\big(\mathscr{W}, \mathscr{W}; \mathfrak{L}(\mathscr{W}, \mathscr{W})\big)$ τ-associated with $\mathscr{V}$, $\tau = \big(1, 1, (1, 1)\big)$).

9.1. *For any $X, Y, Z \in \mathscr{V}$,*

$$(1)\quad (\nabla_X R)_{Y,Z} - R_{X,T(Y,Z)} + \text{cycl} = 0.$$

By 5.1 (c) (or § 5, (3)),

$$(\nabla_X R)_{Y,Z} = \nabla_X R_{Y,Z} - R_{\nabla_X Y, Z} - R_{X, \nabla_X Z}.$$

Consequently, by 7.3,

$$(\nabla_X R)_{Y,Z} + \text{cycl} = (\nabla_X R_{Y,Z} + \text{cycl}) + (-R_{\nabla_X Y,Z} - R_{Y,\nabla_X Z} + \text{cycl})$$
$$= -R_{X,[Y,Z]} + R_{X,\nabla_Y Z} - R_{X,\nabla_Z Y} + \text{cycl} = R_{X,T(Y,Z)} + \text{cycl}.$$

It follows from 9.1 that

9.2. *If $G \in \mathfrak{L}(\mathscr{W}, \mathscr{W}; \mathscr{R})$ is symmetric and $\nabla G = 0$, then*

$$(\nabla_X GR)(Y, Z, U, V) - GR\big(X, T(Y, Z), U, V\big) + \text{cycl} = 0.$$

"cycl" concerns here cyclic permutations of X, Y, Z, as previously.

Let T': $\mathfrak{L}\big(\mathscr{V}; \mathfrak{L}(\mathscr{V}; \mathscr{V})\big)$ be defined as follows:

$$T'(X)Y = T(X, Y) \quad \textit{for} \quad X, Y \in \mathscr{V}.$$

It follows directly from 8.5 that

9.3. *If $\mathscr{R}$ is dyadic, then the covariant derivative $\overline{\nabla} = \nabla - \frac{1}{2}T'$ is symmetric.*

The covariant derivative ∇ on $\mathscr{V}$ is said to be *pseudo-Riemannian* if it is symmetric (i.e. $T = 0$) and there exists a scalar product G on $\mathscr{V}$ such that $\nabla G = 0$.

9.4. *Let $\mathscr{R}$ be dyadic. For every scalar product G on $\mathscr{V}$ there exists exactly one covariant derivative ∇ on $\mathscr{V}$ such that ∇ is symmetric and $\nabla G = 0$.*

The only ∇ is called the *pseudo-Riemannian covariant derivative induced on $\mathscr{V}$ by the scalar product G.*

It follows from 8.3, (6), that $\nabla_X Y$ is uniquely determined by G under the above hypotheses. This proves the uniqueness of ∇. On the other hand, the mapping ∇ defined by 8.3 (6), is a covariant derivative satisfying all the conditions required (proof by a simple calculation).

§ 10. The case where $\mathscr{V}$ is a projection of $\mathscr{W}$. In this section we always assume that the condition § 8 (a) is satisfied (i.e. $\mathscr{V}$ is a submodule of $\mathscr{W}$) and, moreover, there is linear projection P of $\mathscr{W}$ onto $\mathscr{V}$, i.e. a linear map P: $\mathscr{W} \to \mathscr{V}$ such that

(1) $$PV = V \quad \text{for every } V \in \mathscr{V}$$

and, consequently,

(2) $$P(W - PW) = 0 \quad \text{for every } W \in \mathscr{W}.$$

The set of all $W \in \mathscr{W}$ such that $PW = 0$ will be denoted by $\mathscr{N}$. By definition, $\mathscr{N}$ is a submodule of $\mathscr{W}$, and $\mathscr{W}$ is the direct sum of $\mathscr{V}$ and $\mathscr{N}$, i.e. every element $W \in \mathscr{W}$ can be uniquely represented as a sum

$$W = W_1 + W_2,$$

where $W_1 \in \mathscr{V}$ and $W_2 \in \mathscr{N}$. Namely, the only W_1 and W_2 are

$$W_1 = PW \quad \text{and} \quad W_2 = W - PW.$$

W_1 is said to be the *tangent component* of W, and W_2 is called the *pseudonormal component* of W.

For any $X, Y \in \mathscr{V}$ let

$$\nabla^*_X Y = P(\nabla_X Y). \tag{3}$$

By definition, ∇^*_X is a linear mapping from $\mathscr{V}$ into $\mathscr{V}$, viz. it is the composite

$$\nabla^*_X = P\nabla_X. \tag{4}$$

10.1. *∇^*_X is a derivative on $\mathscr{V}$ in the direction $X \in \mathscr{V}$. The mapping $\nabla^*: \mathscr{V} \to \mathfrak{D}(\mathscr{V})$ which assigns $\nabla^*_X \in \mathfrak{D}(\mathscr{V})$ to every $X \in \mathscr{V}$ is a covariant derivative on $\mathscr{V}$.*

∇^* is said to be the *covariant derivative induced on $\mathscr{V}$ by ∇ and P*. Since P is uniquely determined by $\mathscr{N}$, we say also that ∇^* is the *covariant derivative induced on $\mathscr{V}$ by ∇ and $\mathscr{N}$.*

By definition, $\nabla^*_X Y$ is the tangent component of $\nabla_X Y$. Let $H(X, Y)$ be the pseudonormal component of $\nabla_X Y$ $(X, Y \in \mathscr{V})$. By definition

$$H(X, Y) = \nabla_X Y - P\nabla_X Y = \nabla_X Y - \nabla^*_X Y \quad \text{for} \quad X, Y \in \mathscr{V}$$

and

$$H(X, Y) \in \mathscr{N}. \tag{4}$$

10.2. *The mapping $H: \mathscr{V} \times \mathscr{V} \to \mathscr{N}$ is linear, i.e. $H \in \mathfrak{L}(\mathscr{V}, \mathscr{V}; \mathscr{N})$.*

Let T be the torsion tensor of ∇, as previously, and let T^* be the torsion tensor of ∇^*. By definition,

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \quad \text{for} \quad X, Y \in \mathscr{V},$$

$$T^*(X, Y) = \nabla^*_X Y - \Delta^*_Y X - [X, Y] \quad \text{for} \quad X, Y \in \mathscr{V},$$

$$T \in \mathfrak{L}(\mathscr{V}, \mathscr{V}; \mathscr{W}), \qquad T^* \in \mathfrak{L}(\mathscr{V}, \mathscr{V}; \mathscr{V}).$$

10.3. *For any $X, Y \in \mathscr{V}$*

$$T(X, Y) = T^*(X, Y) + \big(H(X, Y) - H(Y, X)\big). \tag{5}$$

$T^*(X, Y)$ is the tangent component of $T(X, Y)$, and $H(X, Y) - H(Y, X)$ is the pseudonormal component of $T(X, Y)$.

Consequently, if $T = 0$ (i.e. ∇ is symmetric), then $T^* = 0$ (i.e. ∇^* is symmetric) and

$$H(X, Y) = H(Y, X) \quad \textit{for all } X, Y \in \mathscr{V}$$

(i.e. H is symmetric).

Let R be the curvature tensor of ∇, as previously, let R^* be the curvature tensor of ∇^*, and let $\bar{R} = R^* - R$. By definition,

$$R_{X,Y} W = \nabla_X \nabla_Y W - \nabla_Y \nabla_X W - \nabla_{[X,Y]} W \in \mathscr{W} \quad \text{for} \quad X, Y \in \mathscr{V} \text{ and } W \in \mathscr{W},$$

$$R^*_{X,Y} V = \nabla^*_X \nabla^*_Y V - \nabla^*_Y \nabla^*_X V - \nabla^*_{[X,Y]} V \in \mathscr{V} \quad \text{for} \quad X, Y, V \in \mathscr{V},$$

$$\bar{R}_{X,Y} V = R^*_{X,Y} V - R_{X,Y} V \in \mathscr{W} \quad \text{for} \quad X, Y, V \in \mathscr{V}.$$

10.4. *For any* $X, Y, V \in \mathscr{V}$

(6) $$\bar{R}_{X,Y}V = H([X, Y], V) - \big(\nabla_X H(Y, V) - \nabla_Y H(X, V)\big) - \\ - \big(H(X, \nabla^*_Y V) - H(Y, \nabla^*_X V)\big).$$

Since ∇^*_X is a directional derivative on $\mathscr{V}$, by § 5 we can form the directional derivative $\nabla^*_X L$ (in a direction $X \in \mathscr{V}$) for any $\mathscr{V}$-tensor L.

If $L: \mathscr{W}^n \to \mathscr{R}$ is n-linear and L^* denotes the restriction of L to $\mathscr{V}$, i.e.

$$L^*(V_1, \ldots, V_n) = L(V_1, \ldots, V_n) \quad \text{for} \quad V_1, \ldots, V_n \in \mathscr{V},$$

then by a simple calculation

(7) $$(\nabla^*_X L^*)(V_1, \ldots, V_n) = (\nabla_X L)(V_1, \ldots, V_n) + \\ + L\big(H(X, V_1), V_2, \ldots, V_n\big) + \ldots + L\big(V_1, \ldots, V_{n-1}, H(X, V_n)\big)$$

for any $X, V_1, \ldots, V_n \in \mathscr{V}$.

In particular, if $G \in \mathfrak{L}(\mathscr{W}, \mathscr{W}; \mathscr{R})$ and G^* is the restriction of G to $\mathscr{V}$,

(8) $$G^*(X, Y) = G(X, Y) \quad \text{for} \quad X, Y \in \mathscr{V},$$

then

(9) $$(\nabla^*_X G)(Y, Z) = (\nabla_X G)(Y, Z) + G\big(H(X, Y), Z\big) + G\big(Y, H(X, Z)\big) \\ \text{for} \quad X, Y, Z \in \mathscr{V}.$$

Note that if G is symmetric, so is G^*.

Let $G \in \mathfrak{L}(\mathscr{W}, \mathscr{W}; R)$ be symmetric. According to the notation introduced in § 7, p. 262, let

$$GR(X, Y, U, W) = G(R_{X,Y}U, W) \quad \text{for} \quad X, Y \in \mathscr{V} \text{ and } U, W \in \mathscr{W},$$
$$GR^*(X, Y, Z, V) = G^*R^*(X, Y, Z, V) = G^*(R^*_{X,Y}Z, V), \\ = G(R^*_{X,Y}Z, V) \quad \text{for} \quad X, Y, Z, V \in \mathscr{V},$$
$$G\bar{R}(X, Y, Z, V) = G(\bar{R}_{X,Y}Z, V) \quad \text{for} \quad X, Y, Z, V \in \mathscr{V}.$$

By definition

$$GR \in \mathfrak{L}(\mathscr{V}, \mathscr{V}, \mathscr{W}, \mathscr{W}; \mathscr{R}), \qquad GR^* = G^*R^* \in \mathfrak{L}(\mathscr{V}, \mathscr{V}, \mathscr{V}, \mathscr{V}; \mathscr{R}), \\ G\bar{R} \in \mathfrak{L}(\mathscr{V}, \mathscr{V}, \mathscr{V}, \mathscr{V}; \mathscr{R}),$$

and

$$G\bar{R} = G^*R^* - GR.$$

If $G \in \mathfrak{L}(\mathscr{W}, \mathscr{W}; \mathscr{R})$ is symmetric, we say that $\mathscr{V}$ and $\mathscr{N}$ are *G-orthogonal* (or *normal*) provided

(10) $$G(V, W) = 0 \quad \text{for every} \quad V \in \mathscr{V} \text{ and } W \in \mathscr{N}.$$

10.5. *If $G \in \mathfrak{L}(\mathscr{W}, \mathscr{W}; \mathscr{R})$ is symmetric, $\nabla G = 0$, and $\mathscr{V}$ and $\mathscr{N}$ are G-ortogonal, then*

$$(11) \qquad \nabla^* G^* = 0,$$

(12) *if $X, V \in \mathscr{V}$ and $W \in \mathscr{N}$, then $G(\nabla_X V, W) + G(V, \nabla_X W) = 0$,*

$$(13) \qquad G\bar{R}(X, Y, Z, V) = G\big(H(Y, Z), H(X, V)\big) - G\big(H(X, Z), H(Y, V)\big) \quad \text{for} \quad X, Y, Z, V \in \mathscr{V}.$$

(11) follows from (9), (10) and (4). Differentiating (10) we get (12). Using 10.4, (10), (12) and (4) we obtain (13).

10.6. *If $\mathscr{R}$ is dyadic, $G \in \mathfrak{L}(\mathscr{W}, \mathscr{W}; \mathscr{R})$ is symmetric, ∇ is symmetric, $\nabla G = 0$, $\mathscr{V}$ and $\mathscr{N}$ are G-orthogonal, and G^* is a scalar product on $\mathscr{V}$, then ∇^* is the pseudo-Riemannian covariant derivative induced on $\mathscr{V}$ by G^*.*

In fact, ∇^* is symmetric by 10.3 and $\nabla^* G^* = 0$ by (11). Thus 10.6 follows from 9.2.

§ 11. The case where $\mathscr{N}$ is one-dimensional. In this section we assume that $\mathscr{V}, \mathscr{W}, \mathscr{N}$ has the same meaning as is § 10, and $G \in \mathfrak{L}(\mathscr{W}, \mathscr{W}; \mathscr{R})$ is symmetric, $\nabla G = 0$, and $\mathscr{V}$ and $\mathscr{N}$ are G-orthogonal. Moreover, we assume that there exists an $N \in \mathscr{N}$ such that for every $W \in \mathscr{N}$ there exists exactly one $a \in \mathscr{R}$ such that

$$(1) \qquad W = aN.$$

It is easy to see that the last condition implies that the ring $\mathscr{R}$ has a unit element 1. We shall also assume that

$$(2) \qquad G(N, N) = 1.$$

It follows from (1) that, for every $X, Y \in \mathscr{V}$, there exists a unique element $h(X, Y) \in \mathscr{R}$ such that

$$(3) \qquad H(X, Y) = h(X, Y) \cdot N.$$

It easily follows from 10.2 and 10.3 that

11.1. *The mapping h is linear, i.e. $h \in \mathfrak{L}(\mathscr{V}, \mathscr{V}; \mathscr{R})$. If ∇ is symmetric, then h is symmetric.*

It follows from 10.5 and from (2) and (3) that

$$(4) \qquad G\bar{R}(X, Y, Z, V) = h(Y, Z)h(X, V) - h(X, Z)h(Y, V) \quad \text{for} \quad X, Y, Z, V \in \mathscr{V}.$$

Since $\mathscr{V}$ and $\mathscr{N}$ are G-orthogonal, we have

$$(5) \qquad G(Y, N) = 0 \quad \text{for every } Y \in \mathscr{V}.$$

By differentiation of (5) and (2) we get the Theorem.

11.2. *For any $X, Y \in \mathscr{V}$*

(6) $$G(\nabla_X N, Y) = -h(X, Y).$$

If $\mathscr{R}$ is dyadic, then for every $X \in \mathscr{V}$

(7) $$G(\nabla_X N, N) = 0.$$

Now we shall prove that

11.3. *For any $X, Y, Z \in \mathscr{V}$*

$$GR(X, Y, Z, N) = (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) + h\big(T(X, Y), Z\big).$$

Using 11.2 we obtain

$$G(\nabla_X \nabla_Y Z, N) = \partial_X \partial_Y \big(G(Z, N)\big) + \partial_X h(Y, Z) + h(X, \nabla_Y Z),$$
$$G(\nabla_Y \nabla_X Z, N) = \partial_Y \partial_X \big(G(Z, N)\big) + \partial_Y h(X, Z) + h(Y, \nabla_X Z),$$
$$G(\nabla_{[X, Y]} Z, N) = \partial_{[X, Y]} \big(G(Z, N)\big) + h([X, Y], Z).$$

Hence

$$\begin{aligned} G(R_{X, Y} Z, N) &= \partial_X h(Y, Z) + h(X, \nabla_Y Z) - \partial_Y h(X, Z) - \\ &\quad - h(Y, \nabla_X Z) - h([X, Y], Z) \\ &= (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) + h\big(T(X, Y), Z\big). \end{aligned}$$

Suppose that ∇ is symmetric. We say that an $X \in \mathscr{V}$ is *principal* if $X \neq 0$ and there is an $a \in \mathscr{R}$ such that $\tilde{h}X = a\tilde{G}X$ (for the meaning of $\tilde{h}$ and $\tilde{G}$, see § 1, (4), p. 254). The element a is said to be the *principal curvature* of X.

We say that an $X \in \mathscr{V}$ is *asymptotic* if $X \neq 0$ and $h(X, X) = 0$.

§ 12. The case where $\mathscr{V}$ and $\mathscr{W}$ have bases. Let $\mathscr{W}$ and $\mathscr{V}$ be such as in § 6. Suppose that the ring $\mathscr{R}$ has a unit element 1. Suppose moreover that $\mathscr{V}$ has a basis

(1) $$V_1, \ldots, V_m$$

and $\mathscr{W}$ has a basis

(2) $$W_1, \ldots, W_n.$$

Let $V^1, \ldots, V^m$ be the basis in $\mathscr{V}^*$, dual to (1), and let $W^1, \ldots, W^n$ be the basis in $\mathscr{W}^*$, dual to (2).

Introducing the notation

$$c_{ij}^k = V^k(V_i, V_j), \quad \Gamma_{i,j}^k = W^k(\nabla_{V_i} W_j), \quad T_{ij}^k = W^k\big(T(V_i, V_j)\big),$$
$$R_{ijk}{}^l = W^l(R_{V_i, V_j} W_k), \quad \text{etc.}$$

we can obtain the known scalar formulas for the coordinates of the covariant derivative of a tensor $L: \mathscr{W}^r \times (\mathscr{W}^*)^s \to \mathscr{R}$, for T_{ij}^k, $R_{ijk}{}^l$, etc. The simple calculation is left to the reader.

§ 13. The case where $\mathscr{R}$ is linear. To obtain the full generality we did note assume any hypotheses about the ring $\mathscr{R}$. In applications to differential geometry the ring $\mathscr{R}$ is linear over reals and has the unit element, and the homogeneity of many operations with respect to multiplication by reals is supposed.

It is very easy to include this case to the general theory investigated in this paper. We have only to suppose that $\mathscr{A}$ has the unit element and is linear over an algebraic field $\mathscr{R}$. In the sequel, when speaking about additivity of some expressions we should additionally postulate the homogeneity with respect to the multiplication by elements of R.

§ 14. Set-theoretical models. Suppose now that $\mathscr{R}$ is a set of real functions defined on a set M. We shall consider M as a topological space with the weakest topology such that all functions $a \in \mathscr{R}$ are continuous. For any set $A \subset M$, the symbol $\mathscr{R}_A$ will denote the set of all real functions a defined on A such that for every $p \in A$ there exists a neighbourhood B of p in the topological subspace A and a function $b \in \mathscr{R}$ such that $a|B = b|B$.

By a *differential space* we shall understand any pair $(M, \mathscr{R})$ where M is a non-empty set and $\mathscr{R}$ is a set of real functions defined on M such that $\mathscr{R} = \mathscr{R}_M$ and

(*) if $a_1, \ldots, a_n \in \mathscr{R}$ and f is an infinitely derivable real function defined on n-dimensional Euclidean space, then the superposition $f \circ (a_1, \ldots, a_n)$ belongs to $\mathscr{R}$.

A differential space $(M, \mathscr{R})$ will be always conceived as a topological space with the weakest topology such that all the functions $a \in \mathscr{R}$ are continuous.

If $(M, \mathscr{R})$ is a differential space and A is a non-void subset of M, then $(A, \mathscr{R}_A)$ is also a differential space. $(A, \mathscr{R}_A)$ is said to be a *differential subspace* of $(M, \mathscr{R})$.

The letter R will always denote the algebraic field of all reals.

Let $(M, \mathscr{R})$ be a differential space. By a *tangent vector* to M at a point $p \in M$ we mean any mapping $v\colon \mathscr{R} \to R$ such that v is additive and homogenous with respect to the multiplication by reals and

$$v(ab) = v(a)\cdot b(p) + a(p)\cdot v(b) \quad \text{for} \quad a, b \in \mathscr{R}.$$

Let M_p denote the linear space of all tangent vectors at p, and let $\mathscr{V}_M$ denote the module of all vector fields V on M, i.e. the set of all mappings V defined on M such that $V_p \in M_p$ for every $p \in M$, and for every $a \in \mathscr{R}$ the mapping Va defined on M by the formula

$$Va(p) = V_p(a) \quad \text{for} \quad p \in M$$

belongs to $\mathscr{R}$.

Suppose that $(M_1, \mathscr{R}_1)$ is another differential space. A mapping $f\colon M \to M_1$ is said to be *differentiable* if $a \circ f \in \mathscr{R}$ for every $a \in \mathscr{R}_1$. If f is a one-to-one mapping from M onto M_1 and both f and f^{-1} are differentiable, f is said to be a *diffeomorphism*, $(M, \mathscr{R})$ and $(M_1, \mathscr{R}_1)$ are said to be *diffeomorphic* provided there exists a diffeomorphism from M onto M_1.

The simplest example of a differential space is given by the pair $(R^n, \mathscr{E})$ where $\mathscr{E}$ is the set of all infinitely derivable real functions defined on the n-dimensional Eulidean space R^n. According to a general notation introduced at the beginning of this section, if O is a non-empty open subset of R^n, then $\mathscr{E}_O$ is the set of all infinitely derivable real functions defined on O. The pair $(O, \mathscr{E}_O)$ is also an example of a differential space.

A differential space $(M, \mathscr{R})$ is said to be an *m-dimensional manifold* (of class C_∞) provided every point $p \in M$ has a neighbourhood A such that $(A, \mathscr{R}_A)$ is diffeomorphic to $(R^m, \mathscr{E})$ (or, equivalently, diffeomorphic to $(O, \mathscr{E}_O)$, where O is an open subset of R^m).

$(M, \mathscr{R})$ is said to be an *m-dimensional submanifold* of an n-dimensional manifold $(M_1, \mathscr{R}_1)$ provided $(M, \mathscr{R})$ is an m-dimensional manifold and $(M, \mathscr{R})$ is a differential subspace of $(M_1, \mathscr{R}_1)$.

Let $(M, \mathscr{R})$ be a differential space. Assuming $\mathscr{V} = \mathscr{W} = \mathscr{V}_M$ in the theory developed in §§ 2-12 (with a modification from § 13) we get a model of the theory. In the case where $(M, \mathscr{R})$ is a manifolds, we get a part of intrinsic differential geometry of manifolds.

Let $(M, \mathscr{R})$ be a differential subspace of a differential space $(N, \mathscr{S})$. Let $\mathscr{W}_{M,N}$ denote the module of all vector fields tangent to N but defined only on M. In other words, elements of $W_{M,N}$ are mappings W defined on M such that $W_p \in N_p$ for every $p \in M$, and for every $a \in \mathscr{S}$ the function Wa defined by the equation

$$Wa(p) = W_p(a) \quad \text{for} \quad p \in M$$

belongs to $\mathscr{R}$. Assuming $\mathscr{V} = \mathscr{V}_M$ and $\mathscr{W} = \mathscr{W}_{M,N}$ in §§ 2-12 (with a modification from § 13) we get another model of the general theory. In the case where $(M, \mathscr{R})$ is a submanifold of a manifold $(N, \mathscr{S})$, we get a part of differential geometry of submanifolds.

Observe that our general theory has also other models.

For instance, instead to consider the ring $\mathscr{R}$ of functions defined on the whole M, the module of all vector fields defined on the whole M etc., we can localize everything to a fixed point $p_0 \in M$. More precisely, we can replace $\mathscr{R}$ by the ring of all germs at p_0, and similarly for $\mathscr{V}$ and $\mathscr{W}$. On this way we get two local models corresponding to the two models described above. All these models are taken from the classical differential geometry.

An essentially different model of the theory can be constructed as follows. Let F be a non-archimedean ordered algebraic field. Similarly as in the case of Euclidean spaces we can define the notion of infinitely derivable mappings $a: F^m \to F$ and the ring $\mathscr{R}$ (linear over F) of all such functions, which exist in abudance, yields a model of our theory.

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