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ON A CERTAIN FUNCTIONAL EQUATION

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In this paper we shall consider the functional equation

$$(1) \qquad f(p) = \sup_q F\big(p, q, f(T(p, q))\big), \quad f(\Theta) = 0,$$

which is a generalization of that considered by Bellman [1]. We shall give some theorems on the existence and uniqueness of solutions of this equation. A theorem on the stability of solution will be also established. Our results will be obtained by the method of successive approximations, the general concept of which was given by Ważewski [3].

1. We introduce the following

Assumption H_1. 1° The function $F(p, q, x)$ is defined for $p = (p_1, \ldots, p_m)$, $p \in D \subset R^m$, $q \in S \subset R^1$, $x \in R^1$, where D is some region in R^m, containing the element $\Theta = (0, \ldots, 0)$ and S is an arbitrarily fixed subset of R^1.

2° For any $(p, q, x) \in D \times R^1 \times R^1$ we have $F(p, q, x) \in R^1$.

3° For any $q \in S$ the function $T(p, q)$ is a transformation of D into D.

4° $F(\Theta, q, 0) = 0$ for any $q \in S$, and $F(p, q, 0)$ considered for $q \in S$ and those $p \in D$ for which $\|p\| \leqslant c_1$, where c_1 an arbitrarily fixed positive number, is uniformly bounded.

5° There exists a non-negative function $\omega(u, v)$ defined for $u, v \geqslant 0$, which is non-decreasing and continuous with respect to u and v, fulfils the condition $\omega(u, 0) \equiv 0$, and, moreover, for any $(p, q, x_i) \in D \times R^1 \times R^1$, $i = 1, 2$, we have the inequality

$$(2) \qquad |F(p, q, x_1) - F(p, q, x_2)| \leqslant \omega(\|p\|, |x_1 - x_2|).$$

6° There exists a non-negative and non-decreasing function $a(u)$ defined and continuous for $u \geqslant 0$ and such that

$$(3) \qquad \|T(p, q)\| \leqslant a(\|p\|), \quad a(0) = 0,$$

for $p \in D$ and $q \in S$.

Assumption H_2. 1° There exists a non-negative and non-decreasing solution $\bar{u}(c)$ of the equation

$$u(c) = \omega\big(c, u(a(c))\big) + v(c), \qquad u(0) = u(0+) = 0, \tag{4}$$

defined for $c \geqslant 0$, where

$$v(c) = \sup_{\|p\| \leqslant c} \sup_{q} |F(p, q, 0)|, \qquad p \in D,\ q \in S,\ c \geqslant 0.$$

2° In the class of functions satisfying the condition $0 \leqslant u(c) \leqslant \bar{u}(c)$ the function $u(c) \equiv 0$ is the only solution of the equation

$$u(c) = \omega\big(c, u(a(c))\big). \tag{5}$$

By Bellman's argument [1] (p. 144-145) one can obtain the following

LEMMA 1. *If*

$$g_i(p) = \sup_q F_i\big(p, q, f_i(T(p, q))\big), \qquad i = 1, 2,$$

then

$$|g_1(p) - g_2(p)| \leqslant \Big|\sup_q F_1\big(p, q, f_1(T(p, q))\big) - F_2\big(p, q, f_2(T(p, q))\big)\Big|.$$

Now let us construct a sequence $\{u_n(c)\}$ by the relations

$$u_0(c) = \bar{u}(c), \qquad c \geqslant 0, \tag{6}$$

$$u_{n+1}(c) = \omega\big(c, u_n(a(c))\big), \qquad n = 0, 1, \ldots, c \geqslant 0.$$

We have

LEMMA 2. *If the assumptions* H_1, 5° *and* 6°, *and* H_2 *are satisfied, then*

$$0 \leqslant u_{n+1}(c) \leqslant u_n(c) \leqslant \bar{u}(c), \qquad n = 0, 1, \ldots, c \geqslant 0, \tag{7}$$

$$\lim_{n \to \infty} u_n(c) = 0, \qquad c \geqslant 0,$$

and the convergence is uniform in each bounded set.

Proof. From relations (4) and (6) we get

$$u_1(c) = \omega\big(c, u_0(a(c))\big) = \omega\big(c, \bar{u}(a(c))\big)$$
$$\leqslant \omega\big(c, \bar{u}(a(c))\big) + v(c) = \bar{u}(c) = u_0(c).$$

Further, we obtain (7) by induction. But (7) implies the convergence of the sequence $\{u_n(c)\}$ to some non-negative function $w(c)$. According to the continuity property of the function $\omega(u, v)$ the function $w(c)$ satisfies equation (5). Now from Assumption H_2, 2°, it follows that $w(c) \equiv 0$.

The uniform convergence of the sequence $\{u_n(c)\}$ follows from the monotonicity property of that sequence and of all functions $u_n(c)$.

2. We shall solve equation (1) by the method of successive approximations. Put

$$(8)\qquad \begin{aligned} &f_0(p) \equiv 0, \quad p \in D, \\ &f_{n+1}(p) = \sup_q F\big(p, q, f_n(T(p,q))\big), \quad p \in D,\ n = 0, 1, \ldots \end{aligned}$$

We have

LEMMA 3. *If assumptions* H_1 *and* H_2 *are satisfied, then*

$$(9)\qquad \sup_{\|p\| \leqslant c} |f_n(p)| \leqslant \bar{u}(c), \quad p \in D,\ n = 0, 1, \ldots,$$

where $\bar{u}(c)$ *is the solution of equation* (4).

Proof. We see that

$$\sup_{\|p\| \leqslant c} |f_0(p)| \equiv 0 \leqslant \bar{u}(c).$$

Further, if we suppose that

$$\sup_{\|p\| \leqslant c} |f_n(p)| \leqslant \bar{u}(c), \quad p \in D,$$

then

$$\begin{aligned} |f_{n+1}(p)| &= \Big|\sup_q F\big(p, q, f_n(T(p,q))\big)\Big| \\ &= \Big|\sup_q \big\{\big[F\big(p, q, f_n(T(p,q))\big) - F(p, q, 0)\big] + F(p, q, 0)\big\}\Big| \\ &\leqslant \sup_q \big[\big|F\big(p, q, f_n(T(p,q))\big) - F(p, q, 0)\big| + |F(p, q, 0)|\big] \\ &\leqslant \sup_q \omega\big(\|p\|, |f_n(T(p,q))|\big) + \sup_q |F(p, q, 0)| \\ &\leqslant \omega\big(c, \bar{u}(a(c))\big) + v(c) = \bar{u}(c). \end{aligned}$$

Hence we get

$$\sup_{\|p\| \leqslant c} |f_{n+1}(p)| \leqslant \bar{u}(c), \quad p \in D.$$

Now, we obtain the assertion of Lemma 3 by induction.

LEMMA 4. *If the assumptions* H_1 *and* H_2 *are satisfied, then*

$$(10)\qquad \sup_{\|p\| \leqslant c} |f_{n+r}(p) - f_n(p)| \leqslant u_n(c)$$

for $p \in D$, $n, r = 0, 1, \ldots,$ *where* $u_n(c)$ *is defined by relation* (6).

Proof. By Lemma 3 we have

$$\sup_{\|p\|\leqslant c} |f_r(p)-f_0(p)| \leqslant \overline{u}(c), \qquad r = 0, 1, \ldots, p \in D.$$

Suppose that

$$\sup_{\|p\|\leqslant c} |f_{r+n}(p)-f_n(p)| \leqslant u_n(c).$$

Take

$$f_{n+r+1}(p) = \sup_q F\Big(p, q, f_{n+r}\big(T(p, q)\big)\Big),$$

$$f_{n+1}(p) = \sup_q F\Big(p, q, f_n\big(T(p, q)\big)\Big).$$

Lemma 1 and inequalities (2) and (3) give

$$\begin{aligned} |f_{n+r+1}(p)-f_{n+1}(p)| & \\ &\leqslant \sup_q \Big| F\Big(p, q, f_{n+r}\big(T(p, q)\big)\Big) - F\Big(p, q, f_n\big(T(p, q)\big)\Big)\Big| \\ &\leqslant \sup_q \omega\Big(\|p\|, \big|f_{n+r}\big(T(p, q)\big)-f_n\big(T(p, q)\big)\big|\Big) \\ &\leqslant \omega\Big(c, u_n\big(a(c)\big)\Big) = u_{n+1}(c). \end{aligned}$$

Hence

$$\sup_{\|p\|\leqslant c} |f_{n+r+1}(p)-f_{n+1}(p)| \leqslant u_{n+1}(c).$$

Finally, we get the assertion of Lemma 4 by induction.

3. Now we can formulate the following

THEOREM 1. *If Assumptions* H_1 *and* H_2 *are satisfied, then there exists a solution* $\bar{f}(p)$ *of equation* (1), *being the limit of the sequence* $\{f_n(p)\}$ *defined by* (8). *The sequence* $\{f_n(p)\}$ *is uniformly convergent in any bounded subset of* D. *The estimations*

$$\sup_{\|p\|\leqslant c} |\bar{f}(p)-f_n(p)| \leqslant u_n(c), \qquad n = 0, 1, \ldots, \tag{11}$$

and

$$\sup_{\|p\|\leqslant c} |\bar{f}(p)| \leqslant \overline{u}(c), \qquad p \in D,\ c \geqslant 0, \tag{12}$$

hold true. The solution $\bar{f}(p)$ *is unique in the class of functions satisfying relation* (12).

Moreover, if $F(p, q, x)$ *and* $T(p, q)$ *are continuous in* $p \in D$ *uniformly with respect to* $q \in S$, *then* $\bar{f}(p)$ *is continuous in* D.

Proof. The uniform convergence of the sequence $\{f_n(p)\}$ follows from (7) and (10) (Lemmas 2 and 4). If r tends to $+\infty$, then (10) gives estimation (11). Estimation (12) is implied by (9). Now we prove that $\bar{f}(p)$ satisfies equation (1). Indeed, for $\|p\| \leqslant c, p \in D$, we have

$$\left|\bar{f}(p) - \sup_q F\left(p, q, \bar{f}(T(p, q))\right)\right|$$

$$\leqslant |\bar{f}(p) - f_{n+1}(p)| + \left|\sup_q F\left(p, q, f_n(T(p, q))\right) - \sup_q F\left(p, q, \bar{f}(T(p, q))\right)\right|$$

$$\leqslant u_{n+1}(c) + \sup_q \omega\left(\|p\|, |f_n(T(p, q)) - \bar{f}(T(p, q))|\right)$$

$$\leqslant u_{n+1}(c) + \omega\left(c, u_n(a(c))\right) \leqslant 2u_{n+1}(c).$$

Now if $n \to \infty$, we get the equation

$$\bar{f}(p) = \sup_q F\left(p, q, \bar{f}(T(p, q))\right),$$

that was to be proved.

To prove that the solution $\bar{f}(p)$ is unique let us suppose that there exists another solution $\bar{\bar{f}}(p)$ such that $\bar{\bar{f}}(p) \not\equiv \bar{f}(p)$ and

$$\sup_{\|p\| \leqslant c} |\bar{\bar{f}}(p)| \leqslant \bar{u}(c).$$

Now we have

$$|f(\bar{\bar{p}}) - f_0(p)| = \left|\sup_q F\left(p, q, \bar{\bar{f}}(T(p, q))\right)\right|$$

$$\leqslant \sup_q \left|F\left(p, q, \bar{\bar{f}}(T(p, q))\right) - F(p, q, 0)\right| + \sup_q |F(p, q, 0|)$$

$$\leqslant \sup_q \omega\left(\|p\|, |\bar{\bar{f}}(T(p, q))|\right) + v(c)$$

$$\leqslant \omega\left(c, \bar{u}\left(a(c)\right)\right) + v(c) = \bar{u}(c).$$

Further, we get

$$\sup_{\|p\| \leqslant c} |\bar{\bar{f}}(p) - f_n(p)| \leqslant u_n(c)$$

by induction and from here it follows that $\bar{\bar{f}}(p) \equiv \bar{f}(p)$. This contradiction proves the uniqueness of $\bar{f}(p)$. Finally, we observe that if $F(p, q, x)$ and $T(p, q)$ are continuous, then all functions $f_n(p)$ are also continuous. Now the continuity of $\bar{f}(p)$ follows from the uniform convergence of the sequence $\{f_n(p)\}$. Thus the proof of Theorem 1 is complete.

Remark. Observe that for proving Theorem 1 it is sufficient to assume that inequality (2) is satisfied only for $|x_i| \leqslant \bar{u}(c)$.

4. We give here a theorem on the uniqueness on the whole of solutions of equation (1).

THEOREM 2. *If assumption* H_1 *is satisfied and the function* $u(c) \equiv 0$ *is the only non-decreasing solution of the inequality*

$$u(c) \leqslant \omega\big(c, u\big(a(c)\big)\big), \qquad u(0) = u(0+) = 0, \tag{13}$$

then equation (1) *has at most one solution.*

Proof. Let us suppose that there exist two solutions $\tilde{f}(p)$ and $\tilde{\tilde{f}}(p)$ of equation (1) such that $\tilde{f}(p) \not\equiv \tilde{\tilde{f}}(p)$. Put

$$m(c) = \sup_{\|p\| \leqslant c} |\tilde{f}(p) - \tilde{\tilde{f}}(p)| \neq 0, \qquad p \in D.$$

By Lemma 1 we get

$$\begin{aligned} |\tilde{f}(p) - \tilde{\tilde{f}}(p)| &\leqslant \sup_q \Big|F\big(p, q, \tilde{f}(T(p, q))\big) - F\big(p, q, \tilde{\tilde{f}}(T(p, q))\big)\Big| \\ &\leqslant \sup_q \omega\big(\|p\|, |\tilde{f}(T(p, q)) - \tilde{\tilde{f}}(T(p, q))|\big) \\ &\leqslant \omega\big(c, m\big(a(c)\big)\big). \end{aligned}$$

Thus we have $m(c) \leqslant \omega\big(c, m\big(a(c)\big)\big)$, and by (13) we conclude that $m(c) \equiv 0$, i.e. $\tilde{f}(p) \equiv \tilde{\tilde{f}}(p)$. This contradiction proves Theorem 2.

5. In order to obtain a theorem on the stability of solutions of equation (1) let us consider the second equation

$$g(p) = \sup_q G\big(p, q, g(T_1(p, q))\big), \qquad g(\theta) = 0, \tag{14}$$

where the functions G and T_1 have the same properties as F and T, as given in assumption H_1.

Suppose that there exists a solution $\bar{g}(p)$ of (14). Set

$$\left\{\begin{aligned} \bar{v}(c) &= \sup_{\|p\| \leqslant c} \sup_q \Big|F\big(p, q, \bar{g}(T(p, q))\big) - G\big(p, q, \bar{g}(T_1(p, q))\big)\Big|, \\ z_0(c) &= \sup_{\|p\| \leqslant c} |\bar{f}(p)| + \sup_{\|p\| \leqslant c} |\bar{g}(p)|, \\ z_{n+1}(c) &= \omega\big(c, z_n\big(a(c)\big)\big) + \bar{v}(c), \qquad n = 0, 1, \ldots \end{aligned}\right. \tag{15}$$

Now we shall prove the following

THEOREM 3. *If assumption* H_1 *is satisfied and*

1° *there exists the limit* $\bar{z}(c)$ *of the sequence* $\{z_n(c)\}$ *as* $n \to \infty$,

2° $\bar{f}(p)$ *and* $\bar{g}(p)$ *are solutions of equations* (1) *and* (14), *respectively, then*

$$\sup_{\|p\| \leqslant c} |\bar{f}(p) - \bar{g}(p)| \leqslant \bar{z}(c), \tag{16}$$

and $\bar{z}(c)$ *is a solution of the equation*

$$u(c) = \omega\big(c, u(a(c))\big) + \bar{v}(c), \qquad c \geqslant 0. \tag{17}$$

Proof. By Lemma 1 we get

$$\begin{aligned}|\bar{f}(p) - \bar{g}(p)| &\leqslant \sup_q \Big|F\big(p, q, \bar{f}(T(p,q))\big) - G\big(p, q, \bar{g}(T_1(p,q))\big)\Big| \\ &\leqslant \sup_q \Big[\Big|F\big(p, q, \bar{f}(T(p,q))\big) - F\big(p, q, \bar{g}(T(p,q))\big)\Big| + \\ &\quad + \Big|F\big(p, q, \bar{g}(T(p,q))\big) - G\big(p, q, \bar{g}(T_1(p,q))\big)\Big|\Big].\end{aligned}$$

Thus for $\|p\| \leqslant c$ we have

$$|\bar{f}(p) - \bar{g}(p)| \leqslant \sup_q \omega\Big(\|p\|, \big|\bar{f}(T(p,q)) - \bar{g}(T(p,q))\big|\Big) + \bar{v}(c).$$

Put

$$z(c) = \sup_{\|p\| \leqslant c} |\bar{f}(p) - \bar{g}(p)|.$$

Now the last inequality implies

$$z(c) \leqslant \omega\big(c, z(a(c))\big) + \bar{v}(c).$$

Whence, in view of the inequality

$$z(c) \leqslant \sup_{\|p\| \leqslant c} |\bar{f}(p)| + \sup_{\|p\| \leqslant c} |\bar{g}(p)| = z_0(c),$$

we get by induction that

$$z(c) \leqslant z_n(c), \qquad n = 0, 1, \ldots$$

Now inequality (16) is implied by the last one as $n \to \infty$.

Remark. If we suppose that (17) has the unique solution $\overline{w}(c)$, then instead of the assumption on the convergence of the sequence $\{z_n(c)\}$ we can suppose that there exists a function $w_0(c)$, $w_0(0) = w_0$, $w(0+) = 0$, satisfying the inequality

$$w_0(c) \geqslant \omega\big(c, w_0(a(c))\big) + \max[\bar{v}(c), z_0(c)].$$

Indeed, we now have $z(c) \leqslant z_0(c) \leqslant w_0(c)$. By setting $w_{n+1}(c) = \omega\big(c, w_n(a(c))\big) + \bar{v}(c)$, $n = 0, 1, \ldots$, we get by induction that $z_{n+1}(c) \leqslant w_{n+1}(c) \leqslant w_n(c) \leqslant w_0(c)$, whence $z(c) \leqslant w_n(c)$, $n = 0, 1, \ldots$ From here, if $n \to \infty$, it follows that $z(c) \leqslant \overline{w}(c)$, where

$$\overline{w}(c) = \lim_{n \to \infty} w_n(c)$$

is the solution of equation (17).

6. Now we are going to consider a special case of the function $\omega(u, v)$. We assume

$$\omega(u, v) = k(u)\cdot v, \quad k(u) \geqslant 0. \tag{18}$$

LEMMA 5. *The condition*

$$\sum_{n=0}^{\infty} k_n(c) v\big(a_n(c)\big) < +\infty, \tag{19}$$

where $a_0(c) = c$ *and* $a_{n+1}(c) = a\big(a_n(c)\big)$, $n = 0, 1, \ldots,$

$$k_0(c) \equiv 1, \quad k_n(c) = \prod_{i=0}^{n-1} k\big(a_i(c)\big), \quad n = 1, 2, \ldots,$$

is necessary and sufficient for the equation

$$u(c) = k(c) u\big(a(c)\big) + v(c) \tag{20}$$

to have a non-negative solution $\bar{u}(c)$.

If (19) *is satisfied and* $\bar{u}(c)$ *is any solution of* (20) *such that*

$$k_n(c)\bar{u}\big(a_n(c)\big) \to 0, \quad n \to \infty,$$

then

$$\bar{u}(c) = \sum_{n=0}^{\infty} k_n(c) v\big(a_n(c)\big). \tag{21}$$

Proof. Necessity. If $\bar{u}(c)$ is any solution of (20), then we get by induction the equations

$$\bar{u}(c) = \sum_{i=0}^{n} k_i(c) v\big(a_i(c)\big) + k_{n+1}(c)\bar{u}\big(a_{n+1}(c)\big), \quad n = 0, 1, \ldots, \tag{22}$$

whence

$$\sum_{i=0}^{n} k_i(c) v\big(a_i(c)\big) \leqslant \bar{u}(c).$$

By letting $n \to \infty$ we get (19).

Equation (21) follows from (22) because the second part of the right-hand side of (22) tends to zero with $n \to \infty$.

Sufficiency of (19) is obvious.

LEMMA 6. *If* $\bar{u}(c)$ *is of the form* (21) *and the function* $u(c)$ *satisfies the inequalities*

$$u(c) \leqslant \bar{u}(c), \quad u(c) \leqslant k(c) u\big(a(c)\big), \tag{23}$$

then $u(c) \equiv 0$.

Proof. From relations (23) we get by induction the following inequalities:

$$u(c) \leqslant k_n(c)\, u\big(a_n(c)\big) \leqslant k_n(c)\, \bar{u}\big(a_n(c)\big), \qquad n = 0, 1, \ldots \tag{24}$$

Because $\bar{u}(c)$ is a solution of (20), equations (22) hold true, and, in view of (21), we have

$$k_n(c)\, \bar{u}\big(a_n(c)\big) \to 0, \qquad n \to \infty.$$

This and (24) imply $u(c) \equiv 0$.

These considerations and Theorem 1 imply

THEOREM 4. *If assumption* H_1 *is satisfied and*

1° $\omega(u, v)$ *has the form* (18), *and*

2° $\sum_{n=0}^{\infty} k_n(c)\, v\big(a_n(c)\big) < +\infty$, *where*

$$v(c) = \sup_{\|p\| \leqslant c} \sup_{q} |F(p, q, 0)|, \qquad p \in D, \ q \in S,$$

then there exists a unique solution $\bar{f}(p)$ *of equation* (1) *with the following properties*:

$$\sup_{\|p\| \leqslant c} |\bar{f}(p)| \leqslant \sum_{n=0}^{\infty} k_n(c)\, v\big(a_n(c)\big),$$

$$\sup_{\|p\| \leqslant c} |\bar{f}(p) - f_n(p)| \leqslant \sum_{i=n}^{\infty} k_i(c)\, v\big(a_i(c)\big).$$

Now Theorem 3 implies

THEOREM 5. *If Assumption* H_1 *is satisfied and*

1° $\omega(u, v)$ *is of the form* (18),

2° $\bar{f}(p)$ *and* $\bar{g}(p)$ *are solutions of equations* (1) *and* (14), *respectively, and*

3° $k_n(c)\, z_0\big(a_n(c)\big) \to 0$ *as* $n \to \infty$,

then

$$\sup_{\|p\| \leqslant c} |\bar{f}(p) - \bar{g}(p)| \leqslant \sum_{n=0}^{\infty} k_n(c)\, \bar{v}\big(a_n(c)\big),$$

where $z_0(c)$ *and* $\bar{v}(c)$ *are defined by relations* (15).

Remark. Both Bellman's theorems ([1], p. 145-148) on the existence and uniqueness of solutions of special cases of equation (1) (the so-called *first* and *second kinds*) are implied by our Theorem 4.

We get the first Bellman's theorem by setting

$$k(c) = 1, \qquad a(c) = ac, \qquad 0 \leqslant a < 1,$$

and the second one by setting

$$k(c) = k, \quad 0 \leqslant k < 1, \quad a(c) = c.$$

But our Theorem 4 is true in many other cases, for instance in the case where

$$v(c) = c, \quad k(c) = k, \quad k \geqslant 0, \quad a(c) = ac, \quad a \geqslant 0, \quad ak < 1.$$

Sobieszek's [2] result can also be obtained from our Theorem 4.

7. Now we shall consider the problem of the uniqueness on the whole of the solutions of equation (1).

Theorem 2 implies the following

THEOREM 6. *If assumption* H_1 *is satisfied and* $\omega(u, v)$ *is of the form* (18) *and one of the conditions*

(a) $k(c) < 1$ *and there exists a* $c_0 \geqslant 0$ *such that* $a(c) < c$ *for* $c \geqslant c_0$,

(b) $k(c) \leqslant 1$ *and* $a(c) < c$ *for any* $c \geqslant 0$,

is satisfied, then there exists at most one solution of equation (1).

Proof. According to Theorem 2, in order to prove Theorem 6 it is sufficient to show that $u(c) \equiv 0$ is the only non-decreasing solution of the inequality

$$u(c) \leqslant k(c)\, u\big(a(c)\big), \tag{25}$$

satisfying the condition $u(0) = u(0+) = 0$.

Let us consider conditions (a) and (b) in turn.

(a) It follows from this condition that for any $c \geqslant 0$ there exists a constant $M(c)$ such that

$$a_n(c) \leqslant M(c), \quad n = 0, 1, \ldots$$

On the other hand, inequality (25) implies that

$$u(c) \leqslant k_n(c)\, u\big(a_n(c)\big), \quad n = 0, 1, \ldots \tag{26}$$

Thus, by the monotonicity properties of $k(c)$ and $u(c)$ we get

$$u(c) \leqslant k_n\big(M(c)\big)\, u\big(M(c)\big), \quad n = 0, 1, \ldots,$$

and, because of the inequality $k(c) < 1$, we finally have $u(c) \equiv 0$ for $c \geqslant 0$.

(b) The inequality $a(c) < c$ gives the relation $\lim\limits_{n\to\infty} a_n(c) = 0$. Inequality (26) yields $u(c) \leqslant u\big(a_n(c)\big)$. This implies $u(c) \equiv 0$ because $u(c) \to 0$ with $c \to 0$. Thus the proof of Theorem 6 is complete.

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