UNIQUENESS OF POSITIVE SOLUTIONS
OF PARABOLIC EQUATIONS WITH UNBOUNDED
COEFFICIENTS

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Introduction. In this note we shall prove the uniqueness of a non-negative solution of the Cauchy problem for the general second order linear parabolic equation with smooth but unbounded coefficients. The uniqueness of non-negative solutions of parabolic equations has been considered in several papers beginning with the well-known work of Widder [9] on the equation of heat conduction. Various extensions of Widder's result to more general parabolic equations can be found in papers [1], [6], [7], and [8]. In all of these papers, however, the coefficients of the equation are assumed to be bounded. Here we shall deal with equations whose coefficients grow at infinity in various ways.

To prove our main result we use several auxiliary theorems. The first of these is a maximum principle for parabolic equations with unbounded coefficients recently proved by Bodanko in paper [5]. Moreover, we use certain properties of the fundamental solution of such equations which were derived in paper [3] and a uniqueness theorem which was proved in paper [2]. It turns out that the required result from paper [2] is not sharp enough in one of the extreme cases for application to the problem which we treat here. In the appendix to this note we prove the appropriate sharpened version of this theorem. The same method also yields an improved version of Bodanko's theorem and of the results of paper [3]. Although these improved results are not actually needed for our work, we include their proofs in the appendix. The key new result which is needed for the proof of our uniqueness theorem is an estimate from below for the fundamental solution which generalizes an estimate derived in [4].

We consider the differential operator

\[ Lu = a_{ij}(x, t) u_{x_i x_j} + b_i(x, t) u_{x_i} + c(x, t) u - u_t \]

in the strip \( S = \mathbb{E}^n \times (0, T] \) for some fixed \( T > 0 \), where \( \mathbb{E}^n \) is the Euclidean
$n$-space. Here and throughout this note we use the usual summation
convention. We make the following assumptions.

(i) The coefficients $a_{ij}, b_i, c$ and their derivatives $a_{ij,x_i}, a_{ij,x_j}, b_{i,x_i}$
are locally Hölder continuous in $\bar{S}$. For some $\lambda$, $0 \leq \lambda \leq 2$, there exist
positive constants $K_1, K_2$ and $K_3$ such that

$$
|a_{ij}| \leq K_1 (|x|^2 + 1)^{(2-\lambda)/2}, \quad |a_{ij,x_i}|, |b_i| \leq K_2 (|x|^2 + 1)^{1/2},
$$

$$
|a_{ij,x_j}|, |b_{i,x_i}|, |c| \leq K_3 (|x|^2 + 1)^{3/2}
$$

for all $(x,t) \in \bar{S}$.

(ii) There exists a constant $\gamma > 0$ such that

$$
a_{ij}(x,t) \beta_i \beta_j \geq \gamma (|x|^2 + 1)^{(2-\lambda)/2} |\beta|^2
$$

for all $(x,t) \in \bar{S}$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n$.

Note that if we write the operator adjoint to $L$ in the form

$$
\tilde{L}u = a_{ij}(\xi, \tau) u_{\xi_i \xi_j} + \tilde{b}_i(\xi, \tau) u_{\xi_i} + \tilde{c}(\xi, \tau) u + u,
$$

then it follows from (i) that

(1) $$
|\tilde{b}_i| \leq \tilde{K}_2 (|\xi|^2 + 1)^{1/2}, \quad |\tilde{c}| \leq \tilde{K}_3 (|\xi|^2 + 1)^{3/2}
$$

for some positive constants $\tilde{K}_2$ and $\tilde{K}_3$.

A function $u = u(x,t)$ is said to be a regular solution of $Lu = 0$
(or $\leq 0$) if $u$ is continuous in $\bar{S} = E^n \times [0, T]$, and if the derivatives
of $u$ which appear in $L$ are continuous and satisfy $Lu = 0$ (or $\leq 0$) at
every point of $S$.

We shall prove the following

**Theorem.** Suppose that the coefficients of $L$ satisfy (i) and (ii). If
$u_1(x,t)$ and $u_2(x,t)$ are two non-negative regular solutions of $Lu = 0$
in $S$ and if $u_1(x,0) = u_2(x,0)$ for all $x \in \mathbb{R}^n$, then $u_1(x,t) = u_2(x,t)$ in $S$.

**Auxiliary results.** We shall use certain results from [2], [3] and [5].
For the convenience of the reader we list them here.

**Theorem B.** Let $\Omega \subseteq \mathbb{R}^n$ be an arbitrary unbounded open domain.
Suppose that the coefficients of $L$ satisfy

$$
a_{ij}(x,t) \beta_i \beta_j \geq 0 \quad \text{for all } (x,t) \in \bar{\Omega} \times [0, T], \quad \beta \in \mathbb{R}^n,
$$

and

$$
|a_{ij}| \leq A(|x|^2 + 1)^{(2-\lambda)/2}, \quad |b_i| \leq B(|x|^2 + 1)^{1/2}, \quad c \leq C(|x|^2 + 1)^{3/2}
$$

in $\bar{\Omega} \times [0, T]$ for some positive constants $A$, $B$, and $C$. If $u$ is a regular
solution of $Lu \leq 0$ in $\Omega \times (0, T]$ such that

$$
u(x,t) \geq 0 \quad \text{for } (x,t) \in (\partial \Omega \times [0, T]) \cup \{\Omega \times (t = 0)\}
$$
and
\[ u(x, t) \geq \begin{cases} 
-M \exp \{ k \log(|x|^2 + 1) + 1 \}^2 & \text{if } \lambda = 0, \\
-M \exp \{ k(|x|^2 + 1)^{\lambda/2} \} & \text{if } 0 < \lambda \leq 2
\end{cases} \]
in \( \Omega \times (0, T] \) for some positive constants \( M \) and \( k \), then \( u(x, t) \geq 0 \) in \( \Omega \times [0, T] \).

Theorem B has been proved in the case \( 0 < \lambda \leq 2 \) by Bodanko in paper [5]. The proof for \( \lambda = 0 \) is given in the appendix to this note. Theorem B is, of course, also valid (with obvious changes) for the adjoint operator \( \tilde{L} \) when condition (i) is satisfied.

The next theorem summarizes some of the results proved in [3]. In the case \( \lambda = 0 \) the results of paper [3] can be sharpened (cf. the appendix). However, the theorem which we quote below suffices for our purposes.

**Theorem AB1.** Suppose that the coefficients of \( L \) satisfy (i) and (ii). Then for some \( T_0 > 0 \) the fundamental solution \( \Gamma(x, t; \xi, \tau) \) of \( Lu = 0 \) exists for all \( x, \xi \in \mathbb{R}^n \), \( 0 < \tau < t \leq T_0 \) and
\[
\int_{\mathbb{R}^n} \Gamma(x, t; \xi, \tau) d\xi \leq M e^{G(x)},
\]
where
\[
G(x) = \begin{cases} 
 k \log(|x|^2 + 1) & \text{if } \lambda = 0, \\
k(|x|^2 + 1)^{\lambda/2} & \text{if } 0 < \lambda \leq 2
\end{cases}
\]
and \( M, k \) are positive constants. Moreover, \( \Gamma(x, t; \xi, \tau) \geq 0 \) and it is the fundamental solution of the adjoint equation \( \tilde{L}u = 0 \) as a function of \( (\xi, \tau) \).

Note that we may assume that \( T \) in the definition of the strip \( S \) is \( \leq T_0 \). For if this is not the case, the proof of the uniqueness theorem can be carried out step by step in the usual way.

**Theorem AB2.** Suppose that the coefficients of \( L \) satisfy (i) and (ii). If \( u = u(x, t) \) is a regular solution of \( Lu = 0 \) in \( S \) such that \( u(x, 0) = 0 \) for \( x \in \mathbb{R}^n \) and
\[
\int_S |u| \exp \left[ - \{ k_0 \log(|x|^2 + 1) + 1 \}^2 \right] dx dt < \infty \quad \text{if } \lambda = 0
\]
or
\[
\int_S |u| \exp \{- k_0 (|x|^2 + 1)^{\lambda/2} \} dx dt < \infty \quad \text{if } 0 < \lambda \leq 2
\]
for some positive constant \( k_0 \), then \( u \equiv 0 \) in \( \overline{S} \).

Theorem AB2 is proved in paper [2] for \( 0 < \lambda \leq 2 \) and in the appendix to this note for \( \lambda = 0 \). We observe that both Theorems AB1 and AB2 hold if the coefficients of \( L \) satisfy (i) and if (ii) is replaced by
the weaker condition (iii) \( a_{ij}(x, t)\beta_i\beta_j \geq \gamma|\beta|^2 \) for \((x, t) \in \mathcal{S}^c\) and \(\beta \in E^n\). The role of condition (ii) will become apparent in the proof of Lemma 2 below.

**Proof of the Theorem.** The proof of our uniqueness theorem is essentially contained in the following three lemmas. The main idea is to show that every non-negative solution of \(Lu = 0\) satisfies the growth condition of Theorem AB2.

**Lemma 1.** If the coefficients of \(L\) satisfy (i) and (ii), and if \(u\) is a non-negative regular solution of \(Lu = 0\), then

\[
u(x, t) \geq \int_{E^n} \Gamma(x, t, \xi, \tau)u(\xi, \tau)d\xi
\]

for all \(x \in E^n, 0 \leq \tau < t \leq T\).

**Proof.** Let \(h_\rho = h_\rho(x)\) be a continuous function defined on \(E^n\) such that \(h_\rho \equiv 1\) for \(|x| < \rho - 1\), \(h_\rho \equiv 0\) for \(|x| \geq \rho\) and \(0 \leq h_\rho \leq 1\), where \(\rho > 1\) is arbitrary. We define

\[
M_\rho = \max_{\mathcal{S}} h_\rho(x)u(x, t)
\]

and consider the function

\[
u_\rho(x, t) = \int_{E^n} \Gamma(x, t; \xi, \tau)h_\rho(\xi,u(\xi, \tau)d\xi
\]

on \(E^n \times (\tau, T]\). It follows from (1) that \(u_\rho(x, t) \leq M_\rho e^{G(\rho)}\). By the definition of fundamental solution (cf. [3])

\[
u_\rho(x, t) \rightarrow h_\rho(\bar{x})u(\bar{x}, \tau) \leq u(\bar{x}, \tau) \quad \text{as} \quad (x, t) \rightarrow (\bar{x}, \tau+).
\]

Moreover, \(u_\rho(x, t)\) is a regular solution of \(Lu = 0\) in \(E^n \times (\tau, T]\). Thus, by Theorem B (with \(\Omega = E^n\)), we obtain \(u(x, t) - u_\rho(x, t) \geq 0\). The assertion of the Lemma now follows from the Lebesgue monotone convergence theorem if we let \(\rho \rightarrow \infty\).

Our next result is an estimate from below for the fundamental solution of \(Lu = 0\).

**Lemma 2.** Let the coefficients of \(L\) satisfy (i) and (ii). Fix \(\bar{x} \in E^n\) and \(t \in (0, T]\). Then for any \(\varepsilon \in (0, t)\) there exist positive constants \(A = A(\varepsilon, \bar{x}), \mu = \mu(\varepsilon, \bar{x})\) such that

\[
\Gamma(\bar{x}, t; \xi, \tau) \geq A \begin{cases} \exp[-\mu \log(|\bar{x} - \xi|^2 + 1)]^2 & \text{if } \lambda = 0, \\ \exp(-\mu |\bar{x} - \xi|^2) & \text{if } 0 < \lambda \leq 2 \end{cases}
\]

for all \((\xi, \tau) \in E^n \times [0, t-\varepsilon]\).

A similar result also holds with \(\xi\) fixed and \(x\) variable. A special case of Lemma 2 and the corresponding result with the roles of \(x\) and \(\xi\)
interchanged was proved in [4]. Note that it is here that we actually use condition (ii). If we assume that $L$ satisfies (i) and (iii) then in place of (3) we would get (by essentially the same proof)

$$
\Gamma(\bar{x}, t; \xi, \tau) \geq \Lambda \exp(-\mu |\bar{x} - \xi|^2)
$$

regardless of the value of $\lambda$. If, for example, $L = A - \partial/\partial t$, then clearly (i) and (iii) are satisfied and the fundamental solution has exactly the order of growth indicated by (4). Thus we cannot expect to get the estimate (3) without a condition stronger than (iii).

**Proof.** Suppose first that $0 < \lambda \leq 2$ and consider the function

$$
V(\bar{x}, t; \xi, \tau) = \exp\left(-\frac{\nu |\bar{x} - \xi|^4}{t - \tau - \epsilon'}\right)
$$

for $(\xi, \tau) \in D \equiv \{|\bar{x} - \xi| > R\} \times [0, t - \epsilon')$, where $\epsilon' = \epsilon/2$, $R = |\bar{x}| + 1$, and $\nu = \nu(\bar{x})$ is a positive constant which we shall choose so that $\tilde{L}V \geq 0$ for $(\xi, \tau) \in D$. Indeed we have

$$
\tilde{L}V = \frac{V}{(t - \tau - \epsilon')^2} \{\nu^2 \lambda^2 |\bar{x} - \xi|^{2\lambda - 4} a_{ij}(\bar{x}_i - \xi_i)(\bar{x}_j - \xi_j) -

- \nu \lambda (\lambda - 2) |\bar{x} - \xi|^{\lambda - 4}(t - \tau - \epsilon') a_{ij}(\bar{x}_i - \xi_i)(\bar{x}_j - \xi_j) -

- \nu \lambda |\bar{x} - \xi|^{\lambda - 4}(t - \tau - \epsilon') a_{ii} + \nu \lambda |\bar{x} - \xi|^{\lambda - 4}(t - \tau - \epsilon') b_i(\bar{x}_i - \xi_i) +

+ \tilde{c}(t - \tau - \epsilon')^2 - \nu |\bar{x} - \xi|^2\}.
$$

Thus in view of (1), (i) and (ii), $|\bar{x} - \xi| \geq R$, and $0 < \lambda \leq 2$ we obtain

$$
\tilde{L}V \geq \frac{V}{(t - \tau - \epsilon')^2} \{\nu^2 \lambda^2 \gamma(|\bar{x}|^2 + 1)^{(2 - \lambda)/2} |\bar{x} - \xi|^{2\lambda - 4} -

- \nu \lambda nTK_1 R^{-1} (|\bar{x}|^2 + 1)^{(2 - \lambda)/2} |\bar{x} - \xi|^{2\lambda - 4} -

- \nu \lambda nTK_2 (|\bar{x}|^2 + 1)^{1/2} |\bar{x} - \xi|^{-1} - T^2 \tilde{K}_2 (|\bar{x}|^2 + 1)^{\lambda/2} - \nu |\bar{x} - \xi|\}
$$

for $(\xi, \tau) \in D$.

Set

$$K_0 = K_0(\bar{x}) = \inf_{\xi \in \mathbb{R}^n} \frac{|\xi|^2 + 1}{|\bar{x} - \xi|^2}.
$$

It is easy to show that $K_0 > 0$, whence

$$
|\xi|^2 + 1 \geq K_0 |\bar{x} - \xi|^2.
$$

On the other hand, since $|\bar{x} - \xi| \geq R = |\bar{x}| + 1$ in $\bar{D}$ we have

$$
(|\bar{x}|^2 + 1)^{\beta/2} \leq (|\xi| + 1)^\beta \leq (|\bar{x} - \xi| + |\bar{x}| + 1)^\beta \leq 2^\beta |\bar{x} - \xi|^\beta
$$

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for any $\beta > 0$. Making use of (7) and (8), it follows from (6) that

$$\tilde{L}V \geq \frac{|\bar{x} - \xi|^4 V}{(t - \tau - \varepsilon')^2} Q(v)$$

for $(\xi, \tau) \in D$, where

$$Q(v) = 2^2 \nu K_0^{\frac{(2^2-1)^2}{2}} v^2 - (2^2-1)n \lambda TK_1 R^{-1} + 2n \lambda T \tilde{K}_2 + 1)v - 2^4 T^3 \tilde{K}_3.$$ 

Since $Q(0) < 0$ it is clear that $Q(v) = 0$ has at least one positive root. Thus if we set $v$ equal to the largest root of $Q(v) = 0$ we have $\tilde{L}V \geq 0$ in $D$.

We define

$$A = A(\varepsilon, \bar{x}) = \min \{t(x, t; \xi, \tau) \mid (\xi, \tau) \in (|\bar{x} - \xi| \leq R) \times [0, t - \varepsilon']\}.$$

It follows from the strong maximum principle that $\Gamma(\bar{x}, t; \xi, \tau) > 0$ (cf. [4]). Thus $A > 0$. Since $\Gamma$ is the fundamental solution of $\tilde{L}u = 0$ as a function of $(\xi, \tau)$, the function

$$z(\xi, \tau) = \Gamma(\bar{x}, t; \xi, \tau) - AV(\bar{x}, t; \xi, \tau)$$

satisfies $\tilde{L}z \leq 0$ in $D$. Moreover, $z \geq 0$ on the set $(|\bar{x} - \xi| = R) \times [0, t - \varepsilon']$ and $\lim z(\xi, \tau) \geq 0$ as $(\xi, \tau) \to (\xi, t - \varepsilon')$ for any $\xi$ such that $|\bar{x} - \xi| \geq R$. Therefore, by Theorem B, we conclude that $z(\xi, \tau) \geq 0$ in $D$ and, in particular, for $(\xi, \tau) \in (|\bar{x} - \xi| \geq R) \times [0, t - \varepsilon]$. In the latter set $t - \tau - \varepsilon' \geq \varepsilon' = \varepsilon/2$ so that we have

$$\Gamma(\bar{x}, t; \xi, \tau) \geq A \exp(-\mu |\bar{x} - \xi|^2),$$

where $\mu = 2\nu/\varepsilon$. Finally, by the definition of $A$ it follows that the estimate (9) holds throughout the strip $B^n \times [0, t - \varepsilon]$.

In case $\lambda = 0$ the proof is quite similar except that the function $V$ defined by (5) is replaced by

$$V(\bar{x}, t; \xi, \tau) = \exp \left[ -\frac{\nu}{t - \tau - \varepsilon'} \{\log((|\bar{x} - \xi|^2 + 1)^2)\} \right].$$

Proceeding as in the case $\lambda > 0$ and using formula (8) together with $\log((|\bar{x} - \xi|^2 + 1) \geq \log 2$ for $|\bar{x} - \xi| \geq R$ and

$$\inf_{|\bar{x} - \xi| \geq 1} \frac{(|\xi|^2 + 1) |\bar{x} - \xi|^2}{(|\bar{x} - \xi|^2 + 1)^2} > 0$$

we obtain the estimate

$$\tilde{L}V \geq \frac{\{\log((|\bar{x} - \xi|^2 + 1)^2)}{(t - \tau - \varepsilon')^2} VQ^*(v),$$

where again $Q^*(v)$ is a quadratic polynomial in $v$ such that $Q^*(0) < 0$ and $Q^*(v) \to +\infty$ as $v \to +\infty$. 
We now apply Lemmas 1 and 2 to prove

**Lemma 3.** Suppose $u$ is a non-negative regular solution of $Lu = 0$ in $S$. Then

$$
\int_0^{T/2} \int_{E^n} u(\xi, \tau) \exp[-\mu \{\log(|\xi|^2 + 1)\}^2] d\xi d\tau < \infty \quad \text{if} \quad \lambda = 0
$$

or

$$
\int_0^{T/2} \int_{E^n} u(\xi, \tau) \exp(-\mu |\xi|^2) d\xi d\tau < \infty \quad \text{if} \quad 0 < \lambda \leq 2,
$$

where $\mu = \mu(T/2, 0)$.

**Proof.** Apply Lemma 1 with $\bar{x} = 0$ and $t = T$ to obtain

$$
\int_{E^n} T(0, T; \xi, \tau) u(\xi, \tau) d\xi \leq u(0, T)
$$

for $\tau \in [0, T)$. Hence, by Lemma 2 with $\bar{x} = 0$, $t = T$ and $\epsilon = T/2$ we have

$$
A \int_{E^n} u(\xi, \tau) \exp[-\mu \{\log(|\xi|^2 + 1)\}^2] d\xi \leq u(0, T) \quad \text{if} \quad \lambda = 0
$$

or

$$
A \int_{E^n} u(\xi, \tau) \exp(-\mu |\xi|^2) d\xi \leq u(0, T) \quad \text{if} \quad 0 < \lambda \leq 2
$$

for $\tau \in [0, T/2]$, where $\mu = \mu(T/2, 0)$. The assertion of the Lemma now follows by integrating these expressions with respect to $\tau$ on $[0, T/2]$.

Our uniqueness theorem is an easy consequence of Lemma 3. Since $u_1$ and $u_2$ are non-negative regular solutions of $Lu = 0$ in $S$ each of them must satisfy the growth condition (10) or (11). Let $v = u_1 - u_2$. Then $v$ is a regular solution of the Cauchy problem $Lv = 0$ in $S$, $v(x, 0) = 0$ for all $x \in E^n$. Moreover, $|v| \leq u_1 + u_2$ implies that $v$ also satisfies the growth condition (10) or (11). It follows from Theorem AB2 that $v = 0$ in $E^n \times [0, T/2]$. The proof of the Theorem is now completed by applying the same argument in the strip $E^n \times [T/2, T]$.

**APPENDIX**

We shall now prove the assertions of Theorems B and AB2 for $\lambda = 0$. We shall also indicate how the results of paper [3] can be sharpened in the case $\lambda = 0$.

**1. Proof of Theorem B.** For arbitrary $\varrho > 0$ and $\beta > 0$ consider the function

$$
v(x, t) = u(x, t) + M \exp[2e^{\beta t} \{k \log(r^2 + 1) + 1\}^2 - \{k \log(\varrho^2 + 1) + 1\}^2],
$$
where \( r = |x| \). It is clear that \( w(x, t) \geq 0 \) for \((x, t) \in \Omega \times [0, T) \cup \{\Omega \times (t = 0)\}\) and for \((x, t) \in \Omega \times [0, T) \cap \{(|x| = \varrho) \times [0, T]\}\). On the other hand, it is easily verified that if \( t \in (0, 1/\beta) \) we have

\[
LW \leq e^{\beta t} W \{k \log (r^2 + 1) + 1\}^2 (E - 2\beta),
\]

where \( W = \exp \left[2e^{\beta t} \{k \log (r^2 + 1) + 1\}^2\right]\) and \( E \) is a positive constant which depends only on \( A, B, C, k \) and \( n \). Thus if we set \( \beta = E \) it follows that \( Lw(0) < 0 \) in \( \Omega \times (0, 1/\beta) \). Now let \((\xi, \tau)\) be any fixed point in \( \Omega \times (0, 1/\beta) \) and assume that \( \varrho > |\xi| \). By the weak maximum principle applied to \( w \in \Omega \times (0, 1/\beta) \cap \{(|x| < \varrho) \times (0, 1/\beta)\} \), \( w(\xi, \tau) \geq 0 \). Therefore if we let \( \varrho \to +\infty \) we obtain \( u(\xi, \tau) \geq 0 \) in the strip \( \Omega \times [0, 1/\beta] \). The proof of Theorem B for \( t > 1/\beta \) can now be carried out in the standard manner.

2. Proof of Theorem AB2. Theorem AB2 is consequence of the following result which is proved in [2]. We consider the operator

\[
Lu = A_{ij}(x, t)u_{x_ix_j} - A_i(x, t)u_{x_i} + A(x, t)u - u_t
\]

for \((x, t) \in S\). Let \( S_\delta = E^n \times (0, \delta) \) and \( \bar{S}_\delta = E^n \times [0, \delta) \) for any \( \delta \in (0, T) \).

**Theorem AB2.** Let \( u \) be such that \( Lu = 0 \) in \( S \) and \( u(x, 0) = 0 \) and let \( \delta \) be a fixed number in \((0, T)\). If there exists a function \( \varphi = \varphi(x, t) \) such that \( \varphi \in C^2(\bar{S}_\delta) \), \( \varphi > 0 \) in every compact subset of \( \bar{S}_\delta \),

\[
L^* \varphi = A_{ij}(x, t)\varphi_{x_ix_j} + A_i(x, t)\varphi_{x_i} + A(x, t)\varphi + \varphi_t \leq 0
\]

in \( \bar{S}_\delta \) and

\[
\int \int_{\bar{S}_\delta} \left| \max_i \sum_j |A_{ij}\varphi_{x_j}| + \varphi (\max_i |A_i| + \max_i |A_i|) \right| dxdt < \infty
\]

then \( u \equiv 0 \) in \( S \).

In view of (1) for \( \lambda = 0 \) it is clear that we can write \( L \) in the form

\[
(12) \quad Lu = A_{ij}(x, t)u_{x_ix_j} - \{A_i(x, t)u_{x_i} + A(x, t)u - u_t\}
\]

for \((x, t) \in E^n \times [0, 1/2\beta)\), where \( k = k_0 + 1 \) and \( \beta > 0 \) is a constant to be specified below. It is easily verified that

\[
L^* \varphi \leq \frac{\varphi}{(1 - \beta t)^2} \{k \log (|x|^2 + 1) + 2\}^2 (E - \beta),
\]

where \( E \) is a positive constant which depends only on \( M_1, M_2, M_3, k \) and \( n \). Thus, if we set \( \beta = 2E \), condition (13) is satisfied in \( E^n \times [0, 1/2\beta] \).
To prove Theorem AB2 we must show that

\[(15) \quad \int\int_{S} |u| \exp\left[-\left\{k_0 \log(|x|^2+1)+1\right\}^2 \right] dx \, dt < \infty \]

implies that (14) is satisfied. It is not difficult to show that

\[
\max_i \sum_j |A_{ij} \varphi_{x_i}| + \varphi (\max_i |A_{ij}| + \max |A_{ij}|) \lesssim (|x|^2+1) F \varphi,
\]

where $F$ is a constant which depends only on $M_1, M_2, k$ and $n$. Since $0 \leq t \leq 1/2\beta$ and $k = k_0 + 1$, we have

\[
\varphi \leq \exp \left[-\left\{k \log(|x|^2+1)+2\right\}^2 \right] \leq (|x|^2+1)^{-1} \exp \left[-\left\{k_0 \log(|x|^2+1)+1\right\}^2 \right]
\]

and hence condition (14) is a consequence of (15).

Theorem AB2 corresponds to Theorem 1 of [2]. By an argument similar to the one given above Theorems 3 and 4 of [2] can also be sharpened in the case $\lambda = 0$. We omit the details.

3. Remarks concerning paper [3]. In paper [3] we prove the existence of a fundamental solution $\Gamma(x, t; \xi, \tau)$ of the equation $\mathcal{L}u = 0$, where

\[
\mathcal{L}u = u_t - \{a_{ij}(x, t)u_{x_i} + a_j(x, t)u_{x_j} - b_j(x, t)u_{x_j} - c(x, t)u \}.
\]

We assume that the coefficients of $\mathcal{L}$ satisfy

\[
\nu |\beta|^2 \leq a_{ij}(x, t) \beta_i \beta_j \leq k_1 (|x|^2 + 1)^{(2-\lambda)/2} |\beta|^2
\]

and

\[
|a_{ij, x_j}|, |a_j|, |b_j| \leq k_2 (|x|^2 + 1)^{1/2}, \quad c + a_{j, j}, c - b_{j, j} \leq k_3 (|x|^2 + 1)^{\lambda/2}.
\]

In particular, we prove that for arbitrary $\alpha > 0$ there exists a $T_a > 0$ such that

\[
\Gamma(x, t; \xi, \tau) = \gamma_\alpha(x, t; \xi, \tau) \exp \{g_\alpha(x, t) - g_\alpha(\xi, \tau)\}
\]

for all $x, \xi \in E^n$ and $0 < \tau < t < T_a$, where $\gamma_\alpha$ is the fundamental solution of an equation related to $\mathcal{L}u = 0$,

\[
\int_{E^n} \gamma_\alpha(x, t; \xi, \tau) d^\xi \leq 1 \quad \text{for all} \quad (x, t) \in E^n \times (\tau, T_a]
\]

and $g_\alpha$ is a function which satisfies

\[(16) \quad C^n \equiv k_1 (|x|^2 + 1)^{(2-\lambda)/2} |\nabla_x g_\alpha|^2 + (2 + n) k_2 (|x|^2 + 1)^{1/2} |\nabla_x g_\alpha| +
+ k_3 (|x|^2 + 1)^{\lambda/2} + |a_{ij}g_{ax_i x_j} - g_{at}| < 0
\]

for all $(x, t) \in E^n \times [0, T_a]$. For proving the existence of the fundamental solution $\Gamma$ it makes no difference how $g_\alpha$ is chosen as long as (16) is satisfied. In paper [3] we also apply the fundamental solution to derive
a representation formula for the solution of the Cauchy problem for $\mathcal{L}$. In this application the choice of $g_\alpha$ is important since it determines the admissible growth classes for the data. For the case $\lambda = 0$ in [3] we used the function

$$g_\alpha(x, t) = \log \frac{|x|^2 + 1}{1 - \beta(a)t},$$

where $\beta(a) > 0$ is a function of $a$ which is determined so that (16) holds. With this choice of $g_\alpha$ we were able to solve the Cauchy problem for data which grows no faster than $\text{const}(|x|^2 + 1)^a$.

We observe that the function

$$g_\alpha(x, t) = \frac{1}{1 - \beta t} \{a \log (|x|^2 + 1) + 1\}^2$$

is a better choice for $g_\alpha$. Indeed, if $t \in [0, 1/2\beta]$ we have

$$C^\alpha \leq \frac{\{a \log (|x|^2 + 1) + 1\}^2}{(1 - \beta t)^2 (E - \beta)},$$

where $E$ is a positive constant which depends only on $a, k_1, k_2, k_3$ and $n$. Thus if we take $\beta(a) = 2E$ and $T_\alpha = 1/2\beta$, then (16) holds. With this choice of $g_\alpha$ the representation formula for the solution of the Cauchy problem given in Theorem III of [3] is valid in the case $\lambda = 0$ for data which grows no faster than

$$\text{const} \exp \{a \log (|x|^2 + 1) + 1\}^2.$$

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