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# ON THE SLIDING OF CURVES

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**1.** In many cases of the study of asymptotical behaviour of differential equations, it is important to know if at some point on the surface of the so-called pipe the sliding of solutions is exterior or interior (see, for instance, Ważewski [2]). This paper gives a sufficient condition for the existence of an exterior (or interior) slide.

**2.** Let $U$ be a domain in the space $(t, x_1, \ldots, x_n)$ homeomorphic with an $(n+1)$-dimensional sphere (here and throughout this paper spheres are open spheres) divided into subdomains $U_1 = U_3$ and $U_2$ by a differentiable hypersurface $F$ homeomorphic with an $n$-dimensional sphere.

Let $\mathscr{R}$ be a family of curves, $\boldsymbol{x}(t) = \big(t, x_1(t), \ldots, x_n(t)\big)$, normal with respect to the axis $Ot$, continuously differentiable, having no end points in the interior of $U$. There is exactly one curve of this family $\mathscr{R}$ through any point of $U$. The curves of family $\mathscr{R}$ and their derivatives depend on initial points of curves in a continuous manner (i.e. if $t^k \to t^0$, $x_i^k(t^k) \to x_i^0(t^0)$, then $x_i^k(t) \to x_i^0(t)$ and $\dot{x}_i^k(t) \to \dot{x}_i^0(t)$ and the convergence is uniform in any closed interval).

The point $\boldsymbol{p}^0 = (t^0, x_1^0, \ldots, x_n^0) \in F$ is called a *point of entry in $U_i$ of the family $\mathscr{R}$* if the curve of the family which satisfies the condition $\boldsymbol{x}(t^0) = \boldsymbol{p}^0$ satisfies also for sufficiently small $\varepsilon > 0$ the conditions

$$\boldsymbol{x}(t) \in \begin{cases} U_{i+1} & \text{for} \quad t \in (t^0 - \varepsilon, t^0), \\ U_i & \text{for} \quad t \in (t^0, t + \varepsilon). \end{cases}$$

If for the curve which satisfies the condition $\boldsymbol{x}(t^0) = \boldsymbol{p}^0 \in F$ we have $\boldsymbol{x}(t) \in U_i$ for $t \in (t^0 - \varepsilon, t^0) \cup (t^0, t^0 + \varepsilon)$, then we call $\boldsymbol{p}^0$ the *point of sliding of the family $\mathscr{R}$ on the side of $U_i$*.

Of course, there can be other types of points beside the entry points in $U_1$, $U_2$ and sliding points on the side of $U_1$ and $U_2$.

Let $\boldsymbol{N}(\boldsymbol{p})$ be the field of unit vectors normal to $F$ (where $\boldsymbol{p} \in F$), directed toward $U_2$. Such a field exists under our differentiability assumption on $F$. Now let $\boldsymbol{p}^0 \epsilon (t^0, x_1^0, \ldots, x_n^0) \epsilon U$ and let $\boldsymbol{x}(t)$ be the curve of the family $\mathscr{R}$ which satisfies the condition $\boldsymbol{x}(t^0) = \boldsymbol{p}^0$. Then we will denote the non-zero vector $[1, \dot{x}_1(t^0), \ldots, \dot{x}_n(t^0)]$ by $\boldsymbol{n}(\boldsymbol{p}^0)$. It will be called the *field-vector.*

The point $\boldsymbol{p}^0 \epsilon F$ is called the *point of tangency of the family* $\mathscr{R}$ if the scalar product $\boldsymbol{n}(\boldsymbol{p}^0) \cdot \boldsymbol{N}(\boldsymbol{p}^0)$ is equal to zero.

Let us further assume that the hypersurface $F$ is divided by a differentiable manifold $C$ homeomorphic with an $(n-1)$-dimensional sphere in two domains in $F$: $F_1$ and $F_2$. We assume that all points of $F_i$ are points of entry of the family $\mathscr{R}$ in the domain $U_i$. Evidently, $\boldsymbol{n}(\boldsymbol{p}) \cdot \boldsymbol{N}(\boldsymbol{p}) \leqslant 0$ for any $\boldsymbol{p} \epsilon F_1$ and $\boldsymbol{n}(\boldsymbol{p}) \cdot \boldsymbol{N}(\boldsymbol{p}) \geqslant 0$ for any $\boldsymbol{p} \epsilon F_2$. By continuity, the points belonging to $C$ are points of tangency of the family $\mathscr{R}$.

**3.** Let $\boldsymbol{p}^0 \epsilon C$ and $\varphi$ be the angle between $\boldsymbol{n}(\boldsymbol{p}^0)$ and the $(n-1)$-dimensional plane which is tangent to $C$ in $\boldsymbol{p}^0$. Let $\varphi \neq 0$. Let $\gamma$ be the angle between $\boldsymbol{n}(\boldsymbol{p}^0)$ and the $Ot$-axis. Obviously, $0 \leqslant \gamma \leqslant \pi/2$. By $S(\varepsilon)$ we denote the common part of the $(n+1)$-dimensional sphere with centre $\boldsymbol{p}^0$ and radius $\varepsilon$ and the sum of rays coming out from $\boldsymbol{p}^0$ and forming with $\boldsymbol{n}(\boldsymbol{p}^0)$ an angle smaller than $\psi/2$, where $\psi > 0$ is smaller than $\varphi$ and $\pi/2-\gamma$.

Under our assumptions there exists an $\varepsilon_1 > 0$ such that if $0 < \varepsilon \leqslant \varepsilon_1$, then either $F_2 \frown S(\varepsilon) \neq 0$ and $F_1 \frown S(\varepsilon) = 0$ or $F_2 \frown S(\varepsilon) = 0$ and $F_1 \frown S(\varepsilon) \neq 0$. In the first case we say that $\boldsymbol{n}(\boldsymbol{p}^0)$ is *directed toward* $F_2$, in the other one that it is *directed toward* $F_1$.

We have the

THEOREM. *If in a point $\boldsymbol{p}^0 \epsilon C$ the vector $\boldsymbol{n}(\boldsymbol{p}^0)$, not tangent to $C$, is directed toward $F_i$, then at $\boldsymbol{p}^0$ the family $\mathscr{R}$ is sliding on the side of $U_i$.*

Proof. For the sake of simplicity let $\boldsymbol{n}(\boldsymbol{p}^0)$ be directed toward $F_1$. Let $\boldsymbol{q}$ be a point lying on a ray coming out from $\boldsymbol{p}^0$ and having the direction and sense of $\boldsymbol{n}(\boldsymbol{p}^0)$. Let $K$ be the common part of $S(\varepsilon)$ and of the hyperplane $P$ which is orthogonal to $\boldsymbol{n}(\boldsymbol{p}^0)$ and contains the point $\boldsymbol{q}$. If $\boldsymbol{q}$ is near enough to $\boldsymbol{p}$, then K will be an $n$-dimensional sphere. Denote by $r$ its radius and by $K_1$ the $(n-1)$-dimensional sphere of radius $r/2$ and centre $\boldsymbol{q}$ contained in the common part of $P$ and the hyperplane tangent to $F$ at $\boldsymbol{p}^0$. Let $K_2$ be the Cartesian product of $K_1$ and of the straight line parallel to $\boldsymbol{N}(\boldsymbol{p}^0)$. Let $T^*(\varepsilon)$ be the common part of $S(\varepsilon)$ and of the set of all points lying on segments beginning at $\boldsymbol{p}^0$ and ending at points of $K_2$.

As $0 < \gamma < \pi/2$, there exists an $\varepsilon_2 \epsilon (0, \varepsilon_1)$ such that the hyperplane $Q: t = t^0 + \varepsilon_2$ has common points with $T^*(\varepsilon)$, but it has no points in common with $K_2 \cdot T^*(\varepsilon)$. Lastly, let $T(\varepsilon)$ be the set of all points $(t, x_1, \ldots, x_n) \epsilon T^*(\varepsilon)$ such that $t \epsilon (t^0, t^0+\varepsilon_2)$.

There exists an $\varepsilon_3 \epsilon (0, \varepsilon_2\rangle$ such that, for any $0 < \varepsilon < \varepsilon_3$, the common part of $T(\varepsilon)$ and $F$ is homeomorphic with an $n$-dimensional sphere. $T(\varepsilon)$

is divided by $F$ into two parts $T_1(\varepsilon)$ and $T_2(\varepsilon)$ contained in $U_1$ and $U_2$ respectively.

The vector $\boldsymbol{n}(\boldsymbol{p}^0)$ is not parallel to the rays lying on the sides of the cone $T(\varepsilon)$. Therefore, by continuity of the field $\boldsymbol{n}(\boldsymbol{p})$, there exists an $\varepsilon_4 \in (0, \varepsilon_3\rangle$ such that all points of the frontier of $T_1(\varepsilon)$ for which $t < t^0 + \varepsilon_4$ are points of entry of the family $\mathscr{R}$ into the interior of $T_1(\varepsilon)$.

If $\boldsymbol{x}(t_1) \in T_1(\varepsilon)$ for a $t_1 \in (t^0, t^0 + \varepsilon_4)$, then also $\boldsymbol{x}(t) \in T_1(\varepsilon)$ for $t \in (t_1, t^0 + \varepsilon_4)$. If this curve had a point common with the complement of $T_1(\varepsilon)$ in $T(\varepsilon)$ for a $t \in (t_1, t^0 + \varepsilon_4)$, then this curve should have a common point with the frontier of $T_1(\varepsilon)$ which would be an entry point into the complement of $T_1(\varepsilon)$, contrary to our assumption that all points of the frontier of $T_1(\varepsilon)$ for which $t \in \langle t^0, t^0 + \varepsilon_4)$ are points of entry of the family $\mathscr{R}$ into the interior of $T_1(\varepsilon)$.

Let $\boldsymbol{p}^k = (t^k, x_1^k, \ldots, x_n^k)$ be a sequence of points such that

1° $\boldsymbol{p}^k \to \boldsymbol{p}^0$ (it follows that $t^k \to t^0$),

2° $\boldsymbol{p}^k$ belongs to the frontier of $T(\varepsilon)$,

3° $\boldsymbol{p}^k \in U_1$,

and let $\boldsymbol{x}^k(t)$ be a curve of the family $\mathscr{R}$ such that $\boldsymbol{x}^k(t^k) = \boldsymbol{p}^k$. Since $\mathscr{R}$ depends continuously on its initial values, we have $\boldsymbol{x}^k(t) \to \boldsymbol{x}^0(t)$, where $\boldsymbol{x}^0(t)$ is that curve of the family $\mathscr{R}$ for which we have $\boldsymbol{x}^0(t^0) = \boldsymbol{p}^0$.

By 2° and 3° $\boldsymbol{p}^k$ is a frontier point of $T_1(\varepsilon)$ and for $t \in (t^0, t^0 + \varepsilon_4)$ any point of the frontier of $T_1(\varepsilon)$ is an entry point. Therefore we must have $\boldsymbol{x}^k(t) \in T_1(\varepsilon)$ for $t \in (t^k, t^0 + \varepsilon_4)$. It follows that $\boldsymbol{x}^0(t) \in T_1(\varepsilon) \cup$ frontier of $T_1(\varepsilon)$. As any point of the frontier of $T_1(\varepsilon)$ with $t \in (t^0, t^0 + \varepsilon_4)$ is a point of entry in $T_1(\varepsilon)$, we have $\boldsymbol{x}^0(t) \in T_1(\varepsilon)$ for all $t \in (t^0, t^0 + \varepsilon_4)$. This means that $\boldsymbol{x}^0(t) \in U_1$ for $t \in (t^0, t^0 + \varepsilon_4)$.

Similarly, we shaw that $\boldsymbol{x}^0(t) \in U_1$ for $t \in (t^0 - \bar{\varepsilon}_4, t^0)$, where $\bar{\varepsilon}_4 > 0$ and $\boldsymbol{p}^0$ is a point of sliding of the family $\mathscr{R}$ on the side of $U_1$. This completes the proof.

**4.** It is possible to prove the Theorem differently, for instance using the surface generated by curves of the family $\mathscr{R}$ containing points of $C$.

With only minor changes in the proof we could replace the assumption that $C$ is continuously differentiable and $\boldsymbol{n}(\boldsymbol{p}^0)$ is not tangent to $C$ by the condition that $C$ is continuous and the contingens of $C$ at $\boldsymbol{p}^0$ does not contain the direction of $\boldsymbol{n}(\boldsymbol{p}^0)$.

The condition of our theorem is sufficient but evidently not necessary. In certain cases it is easy to show that points of tangency are not points of sliding. For instance, if $\boldsymbol{p} \in F$ is a point of tangency and all points in a neighbourhood of it are entry points, then also $\boldsymbol{p}$ is an entry point or, if all points of a $k$-dimensional manifold $S$ lying in $F$ are points of tangency, where $k < n$, then also points of $S$ are entry points.

It follows at once from our theorem that under the assumption of Sections 2 and 3 if $\boldsymbol{n}(\boldsymbol{p})$ is not tangent to $C$ for $\boldsymbol{p} \in C$ and we have a sliding on the side of $U_i$ in one point $\boldsymbol{p}^0 \in C$, then all other points of $C$ are sliding points on the side of $U_i$.

**5.** The theorem of Section 3 can be used to shorten or improve many demonstration where the side of sliding was determined by *ad hoc* methods (see, for instance, Tatarkiewicz [1], p. 39). Some minor changes in the assumptions are sometimes needed in applications.

## REFERENCES

[1] K. Tatarkiewicz, *Sur l'allure asymptotique des solutions de l'équation différentielle du second ordre,* Annales Universitatis Mariae Curie-Skłodowska (A) 7 (1953), p. 19-81.

[2] T. Ważewski, *Sur un principe topologique de l'examen de l'allure asymptotique des intégrales des équations différentielles ordinaires,* Annales de la Société Polonaise de Mathématique 20 (1948), p. 279-313.

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