COLLOQUIUM MATHEMATICUM

VOL. XVII 1967 FASC. 2

REALIZATION OF MAPPINGS AS INVERSE LIMITS

BY

S. K. KAUL (SASKATOON, SASKATCHEWAN)

0. INTRODUCTION

By a *space* we always mean a compact metric space, and by a *mapping* a continuous function. For any finite open covering U of a space X, $N(U)$ denotes the nerve of the covering U ([2], p. 234). For convenience we shall denote the vertex in $N(U)$ corresponding to the member u of U by u itself. The distance $\varrho(x, y)$ between any two points x, y in $N(U)$ is the maximum of the absolute differences of their corresponding barycentric coordinates. That is

$$\varrho(x, y) = \max_{u \in U} |x(u) - y(u)|.$$

For any $x \in X$, the simplex $\sigma(x)$ in $N(U)$ corresponding to all the members of U containing x is called the *carrier* of x. Any mapping a: $X \to N(U)$ such that for $x \in X$, $a(x) \in \operatorname{Int} \sigma(x)$ is called a *barycentric* mapping. Such a mapping exists ([1], p. 175). For any two coverings U and V of X, we write $U < V$ if U refines V.

Suppose a diagram

$$\text{(I)}' \qquad \begin{array}{ccccccccc} \ldots & \xrightarrow{\pi_n} & K_n & \longrightarrow & \ldots & \longrightarrow & K_2 & \xrightarrow{\pi_1} & K_1 \\ & & \downarrow f_n & & & & \downarrow f_2 & & \downarrow f_1 \\ \ldots & \xrightarrow{\psi_n} & L_n & \longrightarrow & \ldots & \longrightarrow & L_2 & \xrightarrow{\psi_1} & L_1 \end{array}$$

of spaces and maps is given. Let K and L be the inverse limits ([2], p. 215) of the inverse systems $\{K_n, \pi_{mn}\}$ and $\{L_n, \psi_{mn}\}$, where for $m > n$, π_{mn}: $K_m \to K_n$ and ψ_{mn}: $L_m \to L_n$ are the composition maps $\pi_n \circ \ldots \circ \pi_{m-1}$ and $\psi_n \circ \ldots \circ \psi_{m-1}$ respectively.

Suppose for any $p = (p_1, p_2, \ldots, p_n, \ldots) \in K$ and any n, $\lim_{k\to\infty} \psi_{kn}[f_k(p_k)]$ exists. Set

$$q_n = \operatorname*{Lim}_{k\to\infty} \psi_{kn}[f_k(p_k)], \qquad n = 1, 2, \ldots,$$

and define

$$f(p) = q = (q_1, q_2, \ldots, q_n, \ldots).$$

$q \in L$. To show this we check that for $m > n$, $\psi_{mn}(q_m) = q_n$. Now by definition

$$q_m = \operatorname*{Lim}_{k\to\infty} \psi_{km}[f_k(p_k)].$$

Since ψ_{mn} is a mapping,

$$\begin{aligned}\psi_{mn}(q_m) &= \psi_{mn}\left[\operatorname*{Lim}_{k\to\infty} \psi_{km}[f_k(p_k)]\right] \\ &= \operatorname*{Lim}_{k\to\infty} \psi_{mn}\left[\psi_{kn}[f_k(p_k)]\right] \\ &= \operatorname*{Lim}_{k\to\infty} \psi_{kn}[f_k(p_k)] = q_n.\end{aligned}$$

Thus f is a well defined function from K to L. We call f the *limit function* of the mappings $\{f_n\}$, and $f: K \to L$ the *inverse limit* of (I)′ (cf. [5]).

The main aim of this paper is to prove the following

THEOREM A. *Let X, Y be spaces and $g: X \to Y$ be any mapping. Then there exists a diagram*

$$\text{(I)}\qquad \begin{array}{ccccccc} \cdots \longrightarrow & N(U_n) & \longrightarrow \cdots \longrightarrow & N(U_2) & \xrightarrow{\pi_1} & N(U_1) \\ & \big\downarrow f_n & & \big\downarrow f_2 & & \big\downarrow f_1 \\ \cdots \longrightarrow & N(V_n) & \longrightarrow \cdots \longrightarrow & N(V_2) & \xrightarrow{\psi_1} & N(V_1) \end{array}$$

where for each positive integer n, $N(U_n)$ and $N(V_n)$ are the nerves of finite open coverings U_n, V_n respectively of X, Y, and f_n is a simplicial mapping such that

(a) *the inverse limit $f: P \to Q$ of* (I) *exists, and*

(b) *the diagram*

$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ \big\downarrow \alpha & & \big\downarrow \beta \\ X & \xrightarrow{g} & Y \end{array}$$

commutes, where α and β are homeomorphisms.

We call $f: P \to Q$ a *realization* as an inverse limit of $g: X \to Y$. Note that the continuity of f follows from (b) above. However, if X and Y are homeomorphic and g is any homeomorphism, the diagram (I) can be so constructed that commutativity holds (Theorem 4).

Remark. In case the diagram (I)′ commutes f can furthermore be shown to be continuous ([2], Theorem 3.13, p. 218). In fact, then the definition of the inverse limit as given here reduces to the usual definition ([2], p. 234). For then, for $k > n$,

$$\psi_{kn}[f_k(p_k)] = f_n[\pi_{kn}(p_k)] = f_n(p_n)$$

and the sequence $\{\psi_{kn}[f_k(p_k)]\}$ is just a constant sequence. In [5] to get the continuity of f a certain "closeness" condition is used ([5], Theorem 2).

1. PRELIMINARIES

(1.1) Auxiliary covering. Let X be a space and $a : X \to K$ be a mapping into a polyhedron K. If K' is a simplicial subdivision of K, then the finite open covering $\{a^{-1}[\mathrm{st}(p)]: p \in K'$ is a vertex of $K'\}$ of X, where $\mathrm{st}(p)$ is the open star of p in K', is called an *auxiliary covering* of X with respect to a and K'.

(1.2) Carrier mapping. Let X, Y be spaces, and $g : X \to Y$ be any mapping. Let U and V be finite open coverings of X and Y respectively. Any mapping $g' : N(U) \to N(V)$ such that for any $x \in X$, the carrier of x in $N(U)$ is mapped by it into the carrier of $g(x)$ in $N(V)$ is called a *carrier mapping* with respect to g. Immediately from this definition we have

LEMMA (1.1). *The composition of any two carrier mappings is again a carrier mapping.*

Let X be a space, U be a finite open covering of X, and $a : X \to N(U)$ be a barycentric mapping. Let $N(U)'$ be any simplicial subdivision of the nerve $N(U)$ of U and U' be the auxiliary covering of X with respect to a and $N(U)'$. The following is then easily established.

LEMMA (1.2). *The correspondence* $a^{-1}\{\mathrm{st}(p_i)] \to p_i$ *for* $\mathrm{st}(p_i) \neq \emptyset$, *where p_i is any vertex in $N(U)'$, defines by linear extension a simplicial isomorphism $i: N(U') \to N(U)'$. Furthermore, $i: N(U') \to N(U)$ is a carrier mapping with respect to the identity mapping on X.*

(1.3). In (1.2) above suppose that $U < g^{-1}[V] = \{g^{-1}[v] : v \in V\}$. For any vertex u in $N(U)$ define a correspondence $u \to v$, where v is a vertex in $N(V)$ such that $g[u] \subset v$. This correspondence takes the vertices of a simplex in $N(U)$ into the vertices of a simplex in $N(V)$ and, therefore, has a linear extension $g' : N(U) \to N(V)$. The simplicial mapping g' is called the *projection mapping* induced by g. It is clearly a carrier mapping.

LEMMA (1.3). *Let σ_p and σ_q be any two simplices with vertices $a_0, \dots, a_p$ and $b_0, \dots, b_q$ respectively. Let $f : \sigma_p \to \sigma_q$ be any linear mapping. Then for any two points x, y in σ_p,*

$$\varrho_2(f(x), f(y)) \leqslant p\, \varrho_1(x, y) \text{ diam } [f(\sigma_p)],$$

where ϱ_1 and ϱ_2 denote the metrics for σ_p and σ_q respectively (see § 0).

Proof. Suppose

$$f(a_i) = \sum_{k=0}^{q} w_{ik} b_k, \quad i = 0, \ldots, p.$$

Then

$$\varrho_2\big(f(a_i), f(a_j)\big) = \max_k |w_{ik} - w_{jk}| \leqslant \operatorname{diam}[f(\sigma_p)].$$

If

$$x = \sum_{i=0}^{p} x_i a_i \quad \text{and} \quad y = \sum_{i=0}^{p} y_i a_i$$

then,

$$\varrho_2\big(f(x), f(y)\big) = \max_k \Big| \sum_{i=0}^{p} x_i w_{ik} - \sum_{i=0}^{p} y_i w_{ik} \Big|, \quad k = 0, \ldots, q,$$

$$= \Big| \sum_{i=0}^{p} x_i w_{il} - \sum_{i=0}^{p} y_i w_{il} \Big| \quad \text{for some } k = l.$$

But, since $\sum_{i=0}^{p} x_i = \sum_{i=0}^{p} y_i = 1$, for any fixed value j, $0 \leqslant j \leqslant k$,

$$\varrho_2\big(f(x), f(y)\big) = \Big| \sum_{i=0}^{p} x_i (w_{il} - w_{jl}) - \sum_{i=0}^{p} y_i (w_{il} - w_{jl}) \Big|$$

$$\leqslant \max |w_{il} - w_{jl}| \sum_{\substack{i=0 \\ i \neq j}}^{p} |x_i - y_i|$$

$$\leqslant \operatorname{diam}[f(\sigma_p)] p\, \varrho_1(x, y).$$

This completes the proof.

The following corollary follows immediately from this lemma:

Corollary (1.1). *If $M \subset \sigma_p$, then*

$$\operatorname{diam}\big[f[M]\big] \leqslant p \operatorname{diam}[M] \operatorname{diam}[f(\sigma_p)].$$

2. REALIZATION OF SPACES

(2.1) Auxiliary inverse system. Let U_n, $n = 1, 2, \ldots$, be finite open coverings of X. The inverse system $\{N(U_n), \pi_{mn}\}$ of the nerves $N(U_n)$ and mappings π_{mn}: $N(U_m) \to N(U_n)$ for $m > n$ is called an *auxiliary inverse system* associated with X if the following conditions are satisfied:

(a) Mesh $U_n \to 0$ as $n \to \omega$.

(b) π_n: $N(U_{n+1}) \to N(U_n)$ is for each n a carrier mapping with respect to the identity on X, and

$$\pi_{mn} = \pi_{m-1} \circ \ldots \circ \pi_n \quad \text{for} \quad m > n.$$

(c) If σ_m denotes any simplex in $N(U_m)$, then, for a given $\varepsilon > 0$, diam $\pi_{mn}(\sigma_m) < \varepsilon$ for m large enough.

(d) For any integer n and $p_n \in N(U_n)$, let $V(p_n)$ denote the intersection of all the members of U_n corresponding to the vertices of the smallest simplex containing p_n. If $\pi_n(p_{n+1}) = p_n$, then $\overline{V(p_{n+1})} \subset V(p_n)$.

(2.2). Suppose an auxiliary inverse system is given for a space X. Let $x \in X$ and $\sigma_n(x)$ denote the carrier of x in $N(U_n)$. From (2.1) (b) it follows that $\{\sigma_n(x), \pi'_{mn}\}$, where $\pi'_n = \pi_n | \sigma_n(x)$, and $\pi'_{mn} = \pi'_{m-1} \circ \ldots \circ \pi'_n$ for $m > n$, is an inverse system. If $\sigma(x)$ denotes the inverse limit of this, and P that of the auxiliary inverse system, then P is compact and $\sigma(x)$ is a compact and non-empty subset of P ([2], Theorem (3.6), p. 217).

LEMMA (2.1). *If x_1 and x_2 are any two distinct points of X, then* $\sigma(x_1) \cap \sigma(x_2) = \emptyset$.

Proof. Since $x_1 \neq x_2$, for sufficiently large integer n_0 no member of U_n, for $n \geqslant n_0$, containing x_1 intersects any of its members containing x_2 (see (2.1) (a)). Hence $\sigma_n(x_1) \cap \sigma_n(x_2) = \emptyset$ for $n \geqslant n_0$. This implies that $\sigma(x_1) \cap \sigma(x_2) = \emptyset$.

(2.3). For any $p \in P$, $p = (p_1, p_2, \ldots)$ and $\pi_n(p_{n+1}) = p_n$, $n = 1, 2, \ldots$ From (2.1) (d), $\overline{V(p_{n+1})} \subset V(p_n)$ for each n, also diam $V(p_n) \to 0$ as $n \to \infty$ ((2.1), (a)). Hence

$$\bigcap_{n=1}^{\infty} V(p_n) = \bigcap_{n=1}^{\infty} \overline{V(p_n)} = \{x\} \subset X.$$

Define

$$a : P \to X$$

by setting for $p \in P$

$$a(p) = x \quad \text{where} \quad \bigcap_{n=1}^{\infty} V(p_n) = \{x\}.$$

LEMMA (2.2). *a is 1-1.*

Proof. Suppose $p = (p_1, p_2, \ldots, p_n, \ldots)$ and $q = (q_1, \ldots, q_n, \ldots)$ are distinct points of P. Then there exists an integer n_0 such that for $n \geqslant n_0$, $p_n \neq q_n$. Let $\varepsilon =$ distance (p_{n_0}, q_{n_0}). From (2.1), (c), there exists an integer m_0 such that for $n \geqslant m_0$, diam $[\pi_{nn_0}(\sigma(p_n))] < \varepsilon$, where $\sigma(p_n)$ is the smallest simplex in $N(U_n)$ containing p_n. Hence $q_n \notin \sigma(p_n)$ for $n \geqslant m_0$. Again, for sufficiently large values of n, $\sigma(p_n)$ and $\sigma(q_n)$ are not faces of the same simplex in $N(U_n)$. For otherwise the simplex σ in $N(U_n)$ containing p_n and q_n for some $n \geqslant m_0$ will satisfy diam $[\pi_{nn_0}(\sigma)] < \varepsilon$, contradicting that the distance $(p_{n_0}, q_{n_0}) = \varepsilon$. Thus for some integer n, $V(p_n) \cap V(q_n) = \emptyset$. Hence

$$\bigcap_{n=1}^{\infty} V(p_n) \neq \bigcap_{n=1}^{\infty} V(q_n),$$

and $a(p) \neq a(q)$.

LEMMA (2.3). *a is onto.*

Proof. For any $x \in X$, $\sigma(x) \neq \emptyset$ (see (2.2)). From the definition of a then for any $p \in \sigma(x)$, $a(p) = x$.

LEMMA (2.4). *a is continuous.*

Proof. Let M be a non-empty closed subset of X. Let Q_n denote the subcomplex of $N(U_n)$ consisting of all the carriers $\sigma_n(x)$ in $N(U_n)$ for $x \in M$, $n = 1, 2, \ldots$ If $\pi'_n = \pi_n | Q_n$, for $n = 1, 2, \ldots$, then since π_n are carrier mappings $\{Q_n, \pi'_{mn}\}$ is an inverse system, where for $m > n$, $\pi'_{mn} = \pi'_{m-1} \circ \ldots \circ \pi'_n$. Since Q_n is a compact and non-empty subset of $N(U_n)$, $Q = \operatorname{Inv\,Lim}\{Q_n, \pi'_{mn}\}$ is a non-empty compact subset of P.

For $x \in M$, $\sigma_n(x) \subset Q_n$, hence $\sigma(x) \subset Q$ and $\bigcup_{x \in M} \sigma(x) \subset Q$.

Let $x \in X - M$. Since M is closed and mesh of U_n is $< 1/2^n$, there exists an integer n_0 such that for $n \geqslant n_0$, no member of U_n containing x intersects any of its members intersecting M. Hence for $n \geqslant n_0$, $\sigma_n(x) \cap Q_n = \emptyset$. This implies that $\sigma(x) \cap Q = \emptyset$, and $Q \subset \bigcup_{x \in M} \sigma(x)$. Hence

$$Q = \bigcup_{x \in M} \sigma(x) = \bigcup_{x \in M} a^{-1}(x) = a^{-1}[M]$$

and a is continuous.

Lemmas (2.2)-(2.4) imply that a is a homeomorphism, and we have

THEOREM 1. *The inverse limit of an auxiliary inverse sequence associated with a space is homeomorphic to it.*

(2.4). Let U_1 and U_2 be finite open coverings of a space X. We say that $\overline{U}_2 < U_1$ if for any $u_2 \in U_2$, $u_2 \subset u_1 \in U_1$, then $\overline{u}_2 \subset u_1$. It is not difficult to see that every finite open covering of a space has such a refinement.

THEOREM 2. *Any space has an auxiliary inverse sequence associated with it.*

Proof. Let X be a space, U_1 be a finite open covering of X of mesh $< \frac{1}{2}$, and $a_1 : X \to N(U_1)$ be a barycentric mapping. Let $N(U)'$ be a simplical subdivision of $N(U)$ such that the mesh of $N(U)'$ is $< 1/2p_1$ where p_1 is the dimension of $N(U_1)$. Let U'_1 be the auxiliary covering of X with respect to a_1 and $N(U_1)'$ (see (1.1)), and i_1: $N(U'_1) \to N(U_1)'$ be the inclusion mapping (Lemma (1.2)).

Let U_2 be a finite open covering of X of mesh $< 1/2^2$ and such that $\bar{U}_2 < U_1'$ (see (2.4)). Let π_1': $N(U_2) \to N(U_1')$ be a projection mapping with respect to the identity on X. Then

$$\pi_1 = i_1 \circ \pi_1' \colon N(U_2) \to N(U_1)$$

is a carrier mapping.

Iterating this process we get an inverse system $\{N(U_n), \pi_{mn}\}$, where for each positive integer n, U_n is a finite open covering of X of mesh $< 1/2^n$; $\bar{U}_{n+1} < U_n'$, where U_n' is the auxiliary covering of X with respect to a barycentric mapping $a_n : X \to N(U_n)$ and $N(U_n)'$, a simplicial subdivision of $N(U_n)$ of mesh $< 1/2p_n$, p_n being the dimension of $N(U_n)$. Furthermore, π_n: $N(U_{n+1}) \to N(U_n)$ is a carrier mapping, and is the composition $i_n \circ \pi_n'$, where π_n': $N(U_{n+1}) \to N(U_n')$ is a projection mapping, and i_n: $N(U_n') \to N(U_n)'$ is the inclusion mapping (Lemma (1.2)).

We claim that $\{N(U_n), \pi_{mn}\}$ is an auxiliary inverse sequence associated with X. Conditions (a) and (b) of (2.1) are clearly satisfied. Condition (d) is a consequence of (2.4). To show that condition (c) is also satisfied, consider a simplex σ_{n+2} in $N(U_{n+2})$. By construction $\pi_{n+1}(\sigma_{n+2}) = \sigma$ is a simplex in $N(U_{n+1})'$ and is of diameter $< 1/2p_{n+1}$. Let σ_{n+1} be the smallest simplex in $N(U_{n+1})$ containing σ. From corollary (1.1),

$$\begin{aligned} \operatorname{diam} \pi_n(\sigma) &\leqslant p_{n+1} \operatorname{diam}(\sigma) \operatorname{diam}[\pi_n(\sigma_{n+1})] \\ &\leqslant p_{n+1} \frac{1}{2p_{n+1}} \operatorname{diam}[\pi_n(\sigma_{n+1})] \\ &\leqslant \frac{1}{2} \cdot \frac{1}{2p_n} = \frac{1}{2^2} \cdot \frac{1}{p_n}. \end{aligned}$$

Hence

$$\operatorname{diam}[\pi_{n+2,n}(\sigma)] \leqslant \frac{1}{2^2} \cdot \frac{1}{p_n}.$$

Iterating this result we get, for $m > n$

$$\operatorname{diam}[\pi_{mn}(\sigma_m)] \leqslant \frac{1}{2^{m-n}} \cdot \frac{1}{p_n}$$

for any simplex σ_m in $N(U_m)$. This implies condition (c) and completes the proof of the theorem.

Note. $\{N(U_n), \pi_{mn}\}$ as constructed above has the property that $\pi_n : N(U_{n+1}) \to N(U_n)$ is linear.

As a consequence of Theorems 1 and 2, we have

THEOREM 3. *Any space can be realized as the inverse limit of an auxiliary inverse sequence associated with it.*

Remark 1. Clearly, the inverse system can be adjusted, if necessary, so that for each positive integer n, dimension of $N(U_n) \leqslant$ dimension of X ([1], Theorem 3.22, p. 188).

Remark 2. It may be noted that Theorem 3, in essence, is a well known result of Freudenthal [3]. In fact, the inverse system constructed in [3] (Satz 1, p. 229) has the additional property that the bounding maps are onto. Extending this result of Freudenthal's completely to compact Hausdorff spaces, Mardešić ([4], Lemma 3, p. 282) has shown that any mapping of a compact Hausdorff space into another can be factored through an inverse limit of an inverse system of polyhedra ([4], Theorem 2, p. 285). However, use of the barycentric mapping to construct an inverse system in Theorem 3, in our case, is basically different from that of Freudenthals. It is also the key to the proofs of Theorems A and 4.

3. *REALIZATION OF MAPPINGS*

Proof of Theorem A. We construct inductively auxiliary inverse systems $\{N(U_n), \pi_{mn}\}$ and $\{N(V_n), \psi_{mn}\}$ associated with X and Y respectively, with the added requirement that for each integer n

$$U_n < g^{-1}[V_n] = \{g^{-1}[v]\colon\ v \in V_n\}.$$

Let $f_n\colon N(U_n) \to N(V_n)$ be a projection mapping with respect to g (see (1.3)). Then we have a diagram

$$\begin{array}{ccccccccc}
\ldots & \xrightarrow{\pi_n} & N(U_n) & \longrightarrow & \ldots & \longrightarrow & N(U_2) & \xrightarrow{\pi_1} & N(U_1) \\
 & & \downarrow{\scriptstyle f_n} & & & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} \\
\ldots & \xrightarrow{\psi_n} & N(V_n) & \longrightarrow & \ldots & \longrightarrow & N(V_1) & \xrightarrow{\psi_1} & N(V_1)
\end{array}$$

Let $P = \mathrm{InvLim}\{N(U_n), \pi_{mn}\}$, $Q = \mathrm{InvLim}\{N(V_n), \psi_{mn}\}$ and $\alpha\colon P \to X$, $\beta\colon Q \to Y$ be homeomorphisms as in Theorem 1. To check that $\{f_n\}$ has a limit f, we must show that for any $p = (p_1, \ldots, p_n, \ldots) \in P$, $\lim\limits_{k\to\omega} \psi_{kn}[f_k(p_k)]$ exists $(n = 1, 2, \ldots)$. Let $\alpha(p) = x$; then from the properties of α,

$$p = \mathrm{InvLim}\{\sigma_n(x), \pi'_{mn}\} = \sigma(x).$$

Hence $p_n \in \sigma_n(x)$. Since f_n is a carrier mapping with respect to g, $f_n(p_n) \in \sigma_n(y)$, where $y = g(x)$, $n = 1, 2, \ldots$ Now for $m > n$, $\psi_{mn}(\sigma_m(y)) \subset \sigma_n(y)$, since ψ's are carrier mappings. Hence $\psi_{mn}[\sigma_m(f_m(p_m))] \subset \sigma_n(y)$. Also for any arbitrary $\varepsilon > 0$, there exists an integer m_0 such that for

$m \geqslant m_0$ diam $\psi_{mn}[\sigma(f_m(p_m))] < \varepsilon$ (see (2.1), (c)). Hence $\{\psi_{mn}[f_m(p_m)]\}$, $m = n+1, n+2, \ldots$, forms a Cauchy sequence lying wholly in $\sigma_n(y)$, and therefore has a limit point q_n in $\sigma_n(y)$ since $\sigma_n(y)$ is compact. Thus $f\colon P \to Q$ is a well defined function (see § 0). This proves Theorem A, (a).

To prove (b), note above that for $n = 1, 2, \ldots, q_n \in \sigma_n(y)$. Hence $q \in \operatorname{Inv Lim}\{\sigma_n(y), \psi'_{mn}\}$. Again, from the definition of the homeomorphism β (α of Theorem 1), $\beta(q) = y$. Thus $\beta[f(p)] = g[\alpha(p)]$ and completes the proof of (b).

THEOREM 4. *Let X, Y be homeomorphic spaces and $g\colon X \to Y$ be any homeomorphism. Then $g\colon X \to Y$ can be realized as an inverse limit of a commutative diagram* (I) (see Theorem A).

Proof. Let U_1 be a finite open covering of X such that U_1 and the open covering $V_1 = g[U_1] = \{g[u]\colon u \in U_1\}$ of Y are both of mesh $< \frac{1}{2}$. Let $f_1\colon N(U_1) \to N(V_1)$ be the projection mapping (see (1.3)) defined by the linear extension of the correspondence $u \to g[u]$ for $u \in U$. Since g is an onto homeomorphism, f_1 is an onto isomorphism. From the definition of the metrics in the nerves (see § 0) f_1 is actually an isometry.

Let $\beta_1\colon Y \to N(V_1)$ be a barycentric mapping. Define $\alpha_1\colon X \to N(U_1)$ by setting

$$\alpha_1 = f_1^{-1} \circ \beta_1 \circ g. \tag{3.1}$$

Since f_1 is an isomorphism, α_1 is well defined. It is easy to check that $\alpha_1^{-1}[\operatorname{st}(u)] = u$ for any $u \in U_1$, hence α_1 is a barycentric mapping.

If r_1 is the dimension of $N(U_1)$, let $N(U_1)'$ be a simplicial subdivision of $N(U_1)$ such that its mesh is $< 1/2r_1$. Since f_1 is an onto isomorphism, corresponding to $N(U_1)'$, it induces a subdivision $N(V_1)'$ of $N(V_1)$, which, since f_1 is an isometry, is also of mesh $< 1/2r_1$. Furthermore, from (3.1), we have,

$$g[\alpha_1^{-1}[\operatorname{st}(p)]] = \beta^{-1}[\operatorname{st}[f(p)]], \tag{3.2}$$

where p is any vertex in $N(U_1)'$ and $f(p) = q$ is a vertex in $N(V_1)'$, and the open stars are taken with respect to the corresponding subdivided complexes. It also follows from (3.1) and (3.2) that

$$V_1' = g[U_1'] = \{g[u']\colon u' \in U_1'\}$$

where U_1' and V_1' are the auxiliary coverings of X and Y with respect to α_1, $N(U_1)'$ and β_1, $N(V_1)'$ respectively. Then the projection mapping

$$f_1'\colon N(U_1') \to N(V_1')$$

defined by the linear extension of the correspondence $\alpha^{-1}[\mathrm{st}(p)] \to \to \beta^{-1}[\mathrm{st}(f_1(p))]$ is an onto isomorphism. Furthermore, the diagram

$$\begin{array}{ccc} N(U_1') & \xrightarrow{i_1} & N(U_1)' \\ \downarrow{\scriptstyle f_1'} & & \downarrow{\scriptstyle f} \\ N(V_1') & \xrightarrow{j_1} & N(V_1)' \end{array}$$

commutes, where i_1 and j_1 are the inclusion mappings (see Lemma (1.2)).

Let U_2 be a finite open covering of X such that U_2 and the finite open covering $V_2 = g[U_2] = \{g[u]:\ u \in U_2\}$ are both of mesh $< 1/2^2$, and further that $\overline{U}_2 < U_1'$ (see (2.4)). Then also $\overline{V}_2 < V_1'$ from (3.2). Let

$$\pi_1':\ N(U_2) \to N(U_1')$$

be any projection mapping with respect to the identity on X, and define

$$\psi_1':\ N(V_2) \to N(V_1')$$

by setting

$$\psi_1' = f_1' \circ \pi_1' \circ f_2^{-1},$$

where

$$f_2:\ N(U_2) \to N(V_2)$$

is the isomorphism defined as f_1 above. It is easy to see then from (3.2) and the definition of f_2 that ψ_1' is a projection mapping with respect to the identity on Y. Setting

$$\psi_1 = j_1 \circ \psi_1' \quad \text{and} \quad \pi_1 = i_1 \circ \pi_1'$$

the following diagram

$$\begin{array}{ccc} N(U_2) & \xrightarrow{\pi_1} & N(U_1) \\ \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} \\ N(V_2) & \xrightarrow{\psi_1} & N(V_1) \end{array}$$

commutes. Iterating the above process we get a commutative diagram

$$\begin{array}{ccccccc} \ldots \xrightarrow{\pi_n} & N(U_n) & \longrightarrow \ldots \longrightarrow & N(U_2) & \xrightarrow{\pi_1} & N(U_1) \\ & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} \\ \ldots \xrightarrow{\psi_n} & N(V_n) & \longrightarrow \ldots \longrightarrow & N(V_2) & \xrightarrow{\psi_1} & N(V_1) \end{array}$$

such that $\{N(U_n), \pi_{mn}\}$ and $\{N(V_n), \psi_{mn}\}$ form auxiliary inverse systems associated with X and Y respectively (see proof of Theorem 2).

It is not difficult to see, following the proof of Theorem A, that the inverse limit of this diagram is a realization of $g\colon X \to Y$.

I wish to express my thanks to Prof. R. E. Chamberlin for suggesting the problem, and to him and Prof. J. H. Case for helpful discussions. The concept of auxiliary covering is due to Prof. Case.

REFERENCES

[1] P. S. Alexandrov, *Combinatorial topology I*, 1956.

[2] S. Eilenberg and N. Steenrod, *Foundations of algebraic topology*, Princeton 1952.

[3] H. Freudenthal, *Entwicklungen von Räumen und ihren Gruppen*, Compositio Mathematica 4 (1936-37), p. 145-234.

[4] S. Mardešić, *On covering dimensions and inverse limits of compact spaces*, Illinois Journal of Mathematics 4 (1960), p. 278-291.

[5] J. Mioduszewski, *Mappings of inverse limits*, Colloquium Mathematicum 10 (1963), p. 39-44.

UNIVERSITY OF SASKATCHEWAN, SASKATOON

Reçu par la Rédaction le 11. 11. 1965