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ON SCHWARZ DIFFERENTIABILITY, III

BY

S. N. MUKHOPADHYAY (BURDWAN, WEST BENGAL)

Let $f(x)$ be a real function defined on a closed interval $[a, b]$ with the understanding that $f(x) = f(a)$ if $x < a$ and $f(x) = f(b)$ if $x > b$. For $x \in [a, b]$, if the limit

$$\lim_{h\to 0} \frac{f(x+h)-f(x-h)}{2h}$$

exists, then it is called the *Schwarz derivative* ([6], p. 36) or the *symmetric derivative* [2] of $f(x)$ at the point x and is denoted by $f^{(1)}(x)$. If the ordinary derivative $f'(x)$ exists, then $f^{(1)}(x)$ also exists and $f^{(1)}(x) = f'(x)$. The converse, however, is not true.

In [5], we have shown that if $f(x)$ is continuous and Schwarz differentiable at each point x in $[a, b]$, then the set of points where $f(x)$ is not differentiable is of the first category. Some other results on Schwarz differentiability can also be found in [1]-[4], [7] and [8].

In the present paper we construct an example of a function which is continuous and Schwarz differentiable everywhere in an interval (a, b) but not differentiable on an everywhere dense set of points there. According to the results already cited (Mukhopadhyay [5]), this everywhere dense set, however, must be a set of the first category.

No generality will be lost if we assume the interval of definition to be $[0, 1]$.

For each positive integer n, let D_n denote the set consisting of the points

$$0, \frac{1}{2^n}, \frac{2}{2^n}, \frac{3}{2^n}, \ldots, \frac{2^n-1}{2^n}, 1.$$

Let

$$u_n(x) = \begin{cases} 2^n\left(x-\dfrac{2k}{2^n}\right), & \text{if } \dfrac{2k}{2^n} \leqslant x \leqslant \dfrac{2k+1}{2^n}, \\ -2^n\left(x-\dfrac{2(k+1)}{2^n}\right), & \text{if } \dfrac{2k+1}{2^n} \leqslant x \leqslant \dfrac{2(k+1)}{2^n} \end{cases}$$

for $k = 0, 1, 2, \ldots, 2^{n-1}-1$ and $n = 1, 2, 3, \ldots$ Then

$$u_n'(x) = \begin{cases} 2^n, & \text{if } \dfrac{2k}{2^n} < x < \dfrac{2k+1}{2^n}, \\ -2^n, & \text{if } \dfrac{2k+1}{2^n} < x < \dfrac{2(k+1)}{2^n}, \end{cases}$$

and $u_n(x)$ is not differentiable at the points of D_n.

Let

$$f(x) = \sum_{i=1}^{\infty} \frac{1}{3^i} u_i(x).$$

Then $f(x)$ will satisfy our requirements in $(0, 1)$. In fact:

I. *$f(x)$ is continuous in $(0, 1)$.*

Since each $u_i(x)$ is continuous and the series $\sum 3^{-i} u_i(x)$ converges uniformly in $(0, 1)$, because $|u_i(x)| \leqslant 1$ for $x \in (0, 1)$ and $i = 1, 2, \ldots$, the result follows immediately.

II. *$f(x)$ is not differentiable at the points of* $\left(\bigcup_1^\infty D_n\right) \frown (0, 1)$.

Let c be any point of $\left(\bigcup_1^\infty D_n\right) \frown (0, 1)$. Then there exists the least positive integer r such that c is a point of D_r but not of D_{r-1}. It, therefore, follows that c is a point of non-differentiability of the functions $u_i(x)$ for $i \geqslant r$, but each of the functions $u_1(x), u_2(x), \ldots, u_{r-1}(x)$ is differentiable at c.

Now

$$f(x) = \sum_{i=1}^{r-1} \frac{1}{3^i} u_i(x) + \sum_{i=r}^{\infty} \frac{1}{3^i} u_i(x) = f_1(x) + f_2(x),$$

say. So, $f_1(x)$ is differentiable at $x = c$. We shall show that $f_2(x)$ is not differentiable at $x = c$. We have

$$\begin{aligned} f_2(x) - f_2(c) &= \sum_{i=r}^{\infty} \frac{1}{3^i} u_i(x) - \sum_{i=r}^{\infty} \frac{1}{3^i} u_i(c) \\ &= \frac{1}{3^r} \{u_r(x) - u_r(c)\} + \sum_{i=r+1}^{\infty} \frac{1}{3^i} u_i(x), \end{aligned}$$

because $u_i(c) = 0$ for $i > r$. So,

$$\frac{f_2(x) - f_2(c)}{x - c} = \frac{1}{3^r} \frac{u_r(x) - u_r(c)}{x - c} + \sum_{i=r+1}^{\infty} \frac{1}{3^i} \frac{u_i(x)}{x - c}. \tag{1}$$

It is now easy to see that the series

$$\sum_{i=r+1}^{\infty} \frac{1}{3^i} \frac{u_i(x)}{x-c}$$

converges uniformly in $0 < |x-c| < \delta$ for some $\delta > 0$. For, if $0 < x-c \leqslant 1/2^i$, then $u_i(x)/(x-c) = 2^i$ (this equality is a consequence of the similarity of two triangles) and if $x-c > 1/2^i$, there is a point x' such that $0 < x'-c \leqslant 1/2^i$, where $u_i(x) = u_i(x')$. So $u_i(x)/(x-c) < u_i(x')/(x'-c) = 2^i$. Hence for $x > c$, $u_i(x)/(x-c) \leqslant 2^i$. Similarly, $u_i(x)/(x-c) \geqslant -2^i$ for $x < c$. This shows that if δ is a suitable fixed positive number, then the series

$$\sum_{i=r+1}^{\infty} \frac{1}{3^i} \frac{u_i(x)}{x-c}$$

converges uniformly in $0 < |x-c| < \delta$.

Taking right-hand and left-hand limits in (1), we get

$$\lim_{x\to c+} \frac{f_2(x)-f_2(c)}{x-c} = -\frac{2^r}{3^r} + \sum_{r+1}^{\infty} \frac{1}{3^i} \cdot 2^i = \frac{2^r}{3^r}$$

and

$$\lim_{x\to c-} \frac{f_2(x)-f_2(c)}{x-c} = \frac{2^r}{3^r} + \sum_{r+1}^{\infty} \frac{1}{3^i} \cdot (-2^i) = -\frac{2^r}{3^r}.$$

So, $f_2(x)$ and, consequently, $f(x)$ is not differentiable at $x = c$. Since c is any point of $\left(\bigcup_1^{\infty} D_n\right) \cap (0,1)$, this proves the assertion.

III. *$f(x)$ is differentiable at every point of* $(0,1)$ *not belonging to* $\bigcup_1^{\infty} D_n$.

Let ξ be any point of $(0,1)$ which does not belong to $\bigcup_1^{\infty} D_n$. Let

$$g_i(h) = \frac{u_i(\xi+h)-u_i(\xi)}{3^i \cdot h}, \qquad \xi+h \in (0,1).$$

Then $|g_i(h)| \leqslant 2^i/3^i$ for every $h \neq 0$. Hence the series $\sum_{i=1}^{\infty} g_i(h)$ is uniformly convergent in $0 < |h| < \delta'$ for some $\delta' > 0$. Thus,

$$\lim_{h\to 0} \frac{f(\xi+h)-f(\xi)}{h} = \lim_{h\to 0} \sum_{i=1}^{\infty} \frac{u_i(\xi+h)-u_i(\xi)}{3^i \cdot h}$$

$$= \sum_{i=1}^{\infty} \lim_{h\to 0} \frac{u_i(\xi+h)-u_i(\xi)}{3^i \cdot h} = \sum_{i=1}^{\infty} \left(\frac{2}{3}\right)^i \eta_i,$$

where $\eta_i = 1$ or -1 according as $2k/2^i < \xi < (2k+1)/2^i$ or $(2k+1)/2^i < \xi < 2(k+1)/2^i$ for some k, where $k = 0, 1, 2, \ldots, 2^{i-1}-1$. This shows that $f(x)$ is differentiable at ξ.

IV. *$f(x)$ is Schwarz differentiable at the points of* $\left(\bigcup_1^\infty D_n\right) \cap (0, 1)$.

Let ξ be any point of $\left(\bigcup_1^\infty D_n\right) \cap (0, 1)$. There exists then the least positive integer r such that $\xi \in D_r$, but $\xi \notin D_{r-1}$. Then

$$\sum_{i=r}^{\infty} \frac{u_i(\xi+h) - u_i(\xi-h)}{3^i \cdot 2h} = 0$$

for $0 < |h| < \delta''$, say. It follows that if

$$f_2(x) = \sum_{i=r}^{\infty} \frac{1}{3^i} u_i(x),$$

then $f_2^{(1)}(\xi) = 0$. The functions $u_1(x), u_2(x), \ldots, u_{r-1}(x)$ are differentiable and, consequently, Schwarz differentiable at ξ. Therefore

$$f_1(x) = \sum_{i=1}^{r-1} \frac{1}{3^i} u_i(x)$$

and $f(x) = f_1(x) + f_2(x)$ is Schwarz differentiable at ξ, too.

Since $\bigcup_1^\infty D_n$ is everywhere dense in $(0, 1)$, the construction of the function $f(x)$ is complete.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BURDWAN

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