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# A STRONGER BRIDGE THEOREM

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A fundamental result in the study of connectivity in bicompact normal spaces is the following, which we shall refer to as

MOORE'S SEPARATION THEOREM. *If $H$ and $K$ are disjoint closed subsets of a bicompact normal space $M$, and no subcontinuum of $M$ intersects both $H$ and $K$, then there exist mutually separated subsets, $A$ and $B$, of $M$ such that $H \subseteq A$, $K \subseteq B$ and $M = A+B$.*

The arguments used to prove Theorem 44 of Chapter 1 of [2] can be applied verbatim to establish this result. It appears that Moore's Separation Theorem is not as well known as it might be. Full use of it would simplify many arguments found in the literature.

In this note, we employ the theorem to sharpen a result of Kowalsky [1]. A subset $X$ of a topological space $S$ is *strongly connected in* $S$ provided that, if $x$ and $y$ are points of $X$, then there is a subcontinuum $K$ of $S$ such that $x \in K$, $y \in K$ and $K \subseteq X$.

THEOREM 1. *Let $H$ and $K$ be disjoint closed subsets of a connected bicompact normal space $M$. Then $M-(H+K)$ contains a subset which is strongly connected in $M$ and whose closure intersects both $H$ and $K$.*

Proof. Let $U_0$ and $V_0$ be open subsets of $M$ such that $H \subseteq U_0$, $K \subseteq V_0$ and $\overline{U}_0 \cap \overline{V}_0 = \emptyset$. Let $\mathscr{U}$ be the collection of open subsets of $M$ which contain $H$ and are contained in $U_0$ and $\mathscr{V}$ the collection of open subsets of $M$ which contain $K$ and are contained in $V_0$. If $U$ and $V$ are members of $\mathscr{U}$ and $\mathscr{V}$, respectively, let $Q(U, V)$ denote the collection of points $x$ in $M-(U_0+V_0)$ such that the component of $M-(U+V)$ which contains $x$ intersects both $\partial U$ and $\partial V$ (the boundaries of $U$ and $V$ respectively).

We first show that $Q(U, V)$ is non-void. $M-(U+V)$ contains a subcontinuum which intersects both $\partial U$ and $\partial V$; otherwise, by Moore's Separation Theorem, there would exist mutually separated sets $A$ and $B$

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such that $\partial U \subseteq A$, $\partial V \subseteq B$ and $M-(U+V) = A+B$, and this implies that $M$ is the sum of the mutually separated sets $A+U$ and $B+V$ which is impossible. A subcontinuum of $M$ which intersects both $\partial U$ and $\partial V$ must contain a point $x$ of $M-(U_0+V_0)$ and, for any such $x$, $x \epsilon Q(U, V)$.

We next show that $Q(U, V)$ is closed in $M$. Let $x$ be a point of $M-(U_0+V_0)$ such that the component $C$ of $M-(U+V)$ containing $x$ does not intersect one of $\partial U$, $\partial V$, say $C \cap \partial U = \emptyset$.

Now $C$ is closed in $M-(U+V)$; so there exist mutually separated sets $A$ and $B$ such that $C \subseteq A$, $\partial U \subseteq B$ and $M-(U+V) = A+B$. For each $y$ in $Q(U, V)$, the component of $M-(U+V)$ which contains $y$ intersects $\partial U$ and therefore is contained in $B$; in particular, $Q(U, V) \subseteq B$ and $x$ is not a limit point of $Q(U, V)$.

Clearly, the collection $\{Q(U, V) | U \epsilon \mathcal{U}, V \epsilon \mathcal{V}\}$ has the finite intersection property. Since $M$ is bicompact, there is a point $x$ in $\bigcap \{Q(U, V) | U \epsilon \mathcal{U}, V \epsilon \mathcal{V}\}$. Let $X(U, V)$ denote the component of $M-(U+V)$ containing $x$ and let $X = \bigcup \{X(U, V) | U \epsilon \mathcal{U}, V \epsilon \mathcal{V}\}$. Then $X$ is strongly connected in $M$, lies in $M-(H+K)$, and $\bar{X}$ intersects both of $H$ and $K$.

Using Theorem 1 and Moore's Separation Theorem it is easy to prove the following stronger form of another theorem in [1]:

THEOREM 2. *Let $H$ and $K$ be disjoint closed subsets of a connected bicompact normal space $M$ and let $Q$ be a closed subset of $M$ containing $H+K$ such that if $C$ is a component of $M-Q$, then $\bar{C} \cap Q = \bar{C} \cap (H+K)$. Then there is a set $X$ which is strongly connected in $M$ and whose closure intersects both $H$ and $K$ such that $X \subseteq M-Q$ or $X \subseteq Q$.*

## *REFERENCES*


[1] H-J. Kowalsky, *Bemerkungen zum Brückensatz*, Abhandlungen der Bayerischen Akademie der Wissenschaften, Mathematisch-Naturwissenschaftliche Klasse, 11 (1963), p. 71-81.

[2] R. L. Moore, *Foundations of point set theory*, American Mathematical Society Colloquium Publications 13, New York 1962 (revised edition).


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