COLLOQUIUM MATHEMATICUM

VOL. XIII 1965 FASC. 2

ON SET-THEORETICALLY INDEPENDENT COLLECTIONS OF BALLS

BY

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Recall that an *atom* of a collection of sets $Z_1, \ldots, Z_k$ in a space X is understood to mean a set of the form $Z_1^{j_1} \cap \ldots \cap Z_k^{j_k}$, where $j_i = \pm 1$ $(i = 1, \ldots, k)$, $Z^1 = Z$, and $Z^{-1} = X \setminus Z$. There are at most 2^k atoms of a collection consisting of k sets, because each atom corresponds to a sequence $j_1, \ldots, j_k$. The collection of sets is said to be *set-theoretically independent* if all its atoms are non-void; then there are exactly 2^k atoms.

In this paper, prepared with kind help of Dr. A. Lelek, we shall give conditions for a finite collection of balls in the n-dimensional Euclidean space E^n to be set-theoretically independent. We also establish some relations between the set-theoretical independence of balls and the linear independence of their centres.

Unless the contrary is explicitly stated a ball will throughout be an open ball. Every ball is determined by a sphere — the boundary of this ball in the Euclidean space. Let us observe that with these definitions each non-void atom of a collection of balls has a non-void interior.

First we shall show some geometrical lemmas. The idea of the proofs of theorems (1) is contained in Lemma 6.

LEMMA 1. *If $H_1, \ldots, H_k$ (where $k \leqslant n$) are $(n-1)$-dimensional hyperplanes in the space E^n and the intersection $H_1 \cap \ldots \cap H_k$ is an $(n-k)$-dimensional hyperplane, then there exists an oblique coordinate system in E^n such that H_i has the equation $x_i = 0$ $(i = 1, \ldots, k)$. In this coordinate system for each component C of $E^n \setminus (H_1 \cup \ldots \cup H_k)$ there exists a sequence $\varepsilon_1, \ldots, \varepsilon_k$ of numbers $\varepsilon_i = \pm 1$ such that C is the set of all points $x \in E^n$ of the form $x = (x_1, \ldots, x_n)$, where $0 \neq x_i = \varepsilon_i |x_i|$ for $i = 1, \ldots, k$.*

Proof. Let us choose vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ in E^n so that

1° $\boldsymbol{a}_i$ is parallel to $\bigcap_{\substack{j=1 \\ j \neq i}}^{k} H_j$ $(i = 1, \ldots, k)$,

(1) This concept, depending on the application of inversions, is due, among others, to A. Ramer and R. Ramer.

2° $\boldsymbol{a}_i$ is not parallel to H_i $(i = 1, \ldots, k)$,

3° $\boldsymbol{a}_{k+1}, \ldots, \boldsymbol{a}_n$ are parallel to $H = H_1 \cap \ldots \cap H_k$ and linearly independent.

Then the vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ are linearly independent. In fact, suppose on the contrary that

$$\beta_1 \boldsymbol{a}_1 + \ldots + \beta_k \boldsymbol{a}_k + \beta_{k+1} \boldsymbol{a}_{k+1} + \ldots + \beta_n \boldsymbol{a}_n = 0$$

and $\beta_1^2 + \ldots + \beta_n^2 > 0$. Thus by condition 3° there exists an index $i \leqslant k$ such that $\beta_i \neq 0$, i. e. the vector $\boldsymbol{a}_i$ can be represented as a linear combination of the vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{i-1}, \boldsymbol{a}_{i+1}, \ldots, \boldsymbol{a}_n$ which are all parallel to H_i, in view of conditions 1° and 3°. This contradicts condition 2°.

Therefore we can construct an oblique coordinate system such that its axes x_i $(i = 1, \ldots, n)$ are parallel to the vectors $\boldsymbol{a}_i$, respectively, and its origin o belongs to H. The point o exists, since $n-k \geqslant 0$ implies $H \neq \emptyset$. The equation $x_i = 0$ determines H_i $(i = 1, \ldots, k)$. Our construction immediately gives the conclusion concerning the components C. It is sufficient to see that each point of $E^n \setminus (H_1 \cup \ldots \cup H_k)$ has the first k coordinates non-vanishing.

LEMMA 2. *If $H_1, \ldots, H_k$, G (where $k \leqslant n$) are $(n-1)$-dimensional hyperplanes in the space E^n, the intersection $H_1 \cap \ldots \cap H_k$ is an $(n-k)$-dimensional hyperplane contained in G, and P_1, P_2 are open half-spaces, on which G cuts E^n, then the set $E^n \setminus (H_1 \cup \ldots \cup H_k)$ has a component contained in P_1, and a component contained in P_2.*

Proof. Let, in the oblique coordinate system from Lemma 1, the hyperplane G have the equation

$$a_1 x_1 + \ldots + a_k x_k + a_{k+1} x_{k+1} + \ldots + a_n x_n = 0.$$

Since $H = H_1 \cap \ldots \cap H_k \subset G$ and the equations

$$x_1 = x_2 = \ldots = x_k = 0$$

determine H, it follows that

$$a_1 x_1 + \ldots + a_k x_k = 0,$$

is also an equation for G, and that $a_1^2 + \ldots + a_k^2 > 0$. We can admit the half-spaces P_1 and P_2 to be given by the inequalities

$$a_1 x_1 + \ldots + a_k x_k < 0 \quad \text{and} \quad a_1 x_1 + \ldots + a_k x_k > 0,$$

respectively. Putting

$$\varepsilon_{ij} = \begin{cases} (-1)^j \operatorname{sign} a_i, & \text{if} \quad a_i \neq 0, \\ +1, & \text{if} \quad a_i = 0, \end{cases}$$

for $i = 1, \ldots, k$, and $j = 1, 2$, we determine the required components by the formula $C_j = \{(x_1, \ldots, x_n)\colon\ 0 \neq x_i = \varepsilon_{ij}|x_i| \text{ for } i = 1, \ldots, k\}$, according to Lemma 1. Obviously, $C_1 \subset P_1$ and $C_2 \subset P_2$.

LEMMA 3. *Let $H_1, \ldots, H_k$ (where $k \leqslant n$) be $(n-1)$-dimensional hyperplanes in the space E^n such that the intersection $H = H_1 \cap \ldots \cap H_k$ is an $(n-k)$-dimensional hyperplane. If K is an $(n-1)$-dimensional hyperplane or a ball in E^n, and $H \cap K = \emptyset$, then the set $E^n \setminus (H_1 \cup \ldots \cup H_k)$ has a component disjoint with the closure $\bar{K}$ of K.*

Proof. If K is a ball, let us construct the half-line L going out from the centre of K and perpendicular to H or — for $H = \{p\}$ being one-point set — passing through p. In the point of intersection of L with the sphere of the ball K let us construct the $(n-1)$-dimensional hyperplane G' perpendicular to L. If K is an $(n-1)$-dimensional hyperplane, let us set $G' = K$. Let G be the $(n-1)$-dimensional hyperplane parallel to G' and containing H. Obviously, G satisfies the conditions of Lemma 2. Since either $\bar{K} \cap P_1 = \emptyset$ or $\bar{K} \cap P_2 = \emptyset$, there exists by Lemma 1 a component of $E^n \setminus (H_1 \cup \ldots \cup H_k)$ that is disjoint with $\bar{K}$.

LEMMA 4. *The union $H_1 \cup \ldots \cup H_k$ of $(n-1)$-dimensional hyperplanes $H_1, \ldots, H_k$ (where $k \leqslant n$) cuts the space E^n on 2^k components if and only if the intersection $H_1 \cap \ldots \cap H_k$ is an $(n-k)$-dimensional hyperplane.*

Proof. We have to prove that the condition formulated in Lemma 4 is (a) sufficient, and (b) necessary.

(a) Let us consider the coordinate system from Lemma 1. Different components of $E^n \setminus (H_1 \cup \ldots \cup H_k)$ correspond to different sequences $\varepsilon_1, \ldots, \varepsilon_k$ of the numbers $\varepsilon_i = \pm 1$. There are 2^k those sequences.

(b) The necessity of the condition will be proved by induction. For one hyperplane it is obvious. Let us suppose that it holds for $k-1$ hyperplanes and consider $(n-1)$-dimensional hyperplanes $H_1, \ldots, H_k$ such that the union $H_1 \cup \ldots \cup H_k$ cuts E^n on 2^k components. Then the union $H_1 \cup \ldots \cup H_{k-1}$ cuts E^n on 2^{k-1} components, because all components of $E^n \setminus (H_1 \cup \ldots \cup H_{k-1})$ are convex, and thus adding H_k to the first $k-1$ hyperplanes the number of components can at most be doubled. By the induction hypothesis, the intersection $H' = H_1 \cap \ldots \cap H_{k-1}$ is an $(n-k+1)$-dimensional hyperplane. Since H_k is an $(n-1)$-dimensional hyperplane, the intersection

$$H_1 \cap \ldots \cap H_k = H' \cap H_k$$

can be 1° the void set, 2° the hyperplane H', or 3° an $(n-k)$-dimensional hyperplane. In the cases 1° and 2° H_k cannot intersect all components of $E^n \setminus (H_1 \cup \ldots \cup H_{k-1})$ according to Lemmas 3 and 2, respectively, hence $H_1 \cup \ldots \cup H_k$ cannot cut E^n on 2^k components. So only the case 3° is possible, which completes the proof.

LEMMA 5. *If the intersection $H_1 \cap \ldots \cap H_n$ of $(n-1)$-dimensional hyperplanes is a one-point set $\{p\}$, then p is an accumulation point of each component of the set $E^n \setminus (H_1 \cup \ldots \cup H_n)$.*

Proof. Take the oblique coordinate system given in Lemma 1. Obviously, p is its origin. Lemma 1 immediately implies that each component of $E^n \setminus (H_1 \cup \ldots \cup H_k)$ intersects every ball with centre at the point p.

LEMMA 6. *Let $Q_1, \ldots, Q_k$ be balls in the space E^n and let $\iota\colon M^n \to M^n$ be an inversion of the Möbius space $M^n \supset E^n$ with centre at a point $s \in S_1 \cap \ldots \cap S_k$, where S_i is the sphere of the ball Q_i $(i = 1, \ldots, k)$. Denote by H_i the $(n-1)$-dimensional hyperplane $\iota(S_i \setminus \{s\})$. If B is a component of the set $E^n \setminus (H_1 \cup \ldots \cup H_k)$, then there exists an atom A of the collection of balls $Q_1, \ldots, Q_k$ in E^n such that*

$$\iota(B \setminus \{s\}) = \operatorname{Int} A \subset A \subset \overline{\iota(B \setminus \{s\})} \cup \{s\},$$

where $\operatorname{Int} Z$ *and* $\bar{Z}$ *denote the interior and the closure of the set Z in E^n, respectively. Taking the atom A for the component B one determines a 1-1 correspondence between the components of the set $E^n \setminus (H_1 \cup \ldots \cup H_k)$ and the non-void atoms of the collection $Q_1, \ldots, Q_k$.*

Proof. The open half-spaces, on which H_i cuts E^n, are the sets $\iota(Q_i)$ and $\iota(E^n \setminus \bar{Q}_i) \cup \{s\}$. Since

$$Q_i = \operatorname{Int} Q_i^1 \quad \text{and} \quad E^n \setminus \bar{Q}_i = \operatorname{Int} Q_i^{-1},$$

there exist numbers $j_i = \pm 1$ $(i = 1, \ldots, k)$ such that

$$\text{(i)} \qquad B \setminus \{s\} = \bigcap_{i=1}^{k} \iota(\operatorname{Int} Q_i^{j_i}) = \iota(\operatorname{Int} \bigcap_{i=1}^{k} Q_i^{j_i}),$$

where the last equality is a consequence of the fact that the inversion ι is 1-1. Moreover, the closed half-spaces containing the component B are the closures of the sets $\iota(\operatorname{Int} Q_i^{j_i})$ in E^n $(i = 1, \ldots, k)$, and the intersection of these closed half-spaces is equal to $\bar{B}$. Observe that every set $\iota(Q_i^{j_i} \setminus \{s\})$ is an open half-space or a closed half-space without the point s.

Hence we obtain

$$\text{(ii)} \qquad \bigcap_{i=1}^{k} \iota(Q_i^{j_i} \setminus \{s\}) \subset \bar{B}.$$

Let $A = Q_i^{j_i} \cap \ldots \cap Q_k^{j_i}$. According to (i), the set $\iota(B \setminus \{s\})$ is the interior of A in E^n. Since the inversion ι transforms $E^n \setminus \{s\}$ homeomorphically onto itself and the closure of a subset $Z \subset E^n \setminus \{s\}$ in the space $E^n \setminus \{s\}$ is equal to $\bar{Z} \setminus \{s\}$, we have

$$\iota(\overline{A\setminus\{s\}}\setminus\{s\}) = \overline{\iota(A\setminus\{s\})}\setminus\{s\} = \iota(\overline{\bigcap_{i=1}^{k} Q_i^{j_i}\setminus\{s\}})\setminus\{s\}$$
$$= \overline{\bigcap_{i=1}^{k} \iota(Q_i^{j_i}\setminus\{s\})}\setminus\{s\} \subset \overline{B}\setminus\{s\} = \overline{B\setminus\{s\}}\setminus\{s\},$$

by virtue of (ii). It follows that

$$A \subset (A\setminus\{s\}) \cup \{s\} \subset (\overline{A\setminus\{s\}}\setminus\{s\}) \cup \{s\}$$
$$\subset \iota(\overline{B\setminus\{s\}}\setminus\{s\}) \cup \{s\} = \overline{\iota(B\setminus\{s\})} \cup \{s\}.$$

In this way we have proved the first conclusion of Lemma 6, containing inclusions. It implies the second conclusion. Indeed, the function under which the atom A corresponds to the component B must be 1-1, for the inversion ι is so. Each non-void atom of the collection $Q_1, \ldots, Q_k$ corresponds to some component B, because this atom has a non-void interior disjoint with $S_1 \cup \ldots \cup S_k$, and the inversion ι is a homeomorphism of the Möbius space M^n onto itself, sending $H_1 \cup \ldots \cup H_k$ into $S_1 \cup \ldots \cup S_k$.

Remark. The point s can be an isolated point of an atom A. It is so e. g. for the atom

$$A = Q_1^{-1} \cap Q_2^{-1} \cap Q_3^{-1}$$

of the collection of three circles Q_1, Q_2, Q_3 on the plane E^2, whose boundaries intersect at a single point s (fig. 1). Thus the symbol $\{s\}$ written down at the end of inclusions in Lemma 6 cannot be removed from this formula.

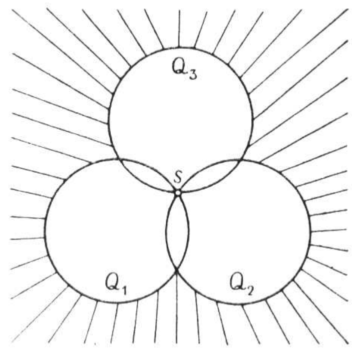

Fig. 1

Lemma 7. *If a collection of balls $Q_1, \ldots, Q_k$ (where $k \leqslant n$) in the space E^n is set-theoretically independent and all its atoms are connected, then for every $(n-1)$-dimensional hyperplane H passing through the centres of the balls $Q_1, \ldots, Q_k$ the collection $H \cap Q_1, \ldots, H \cap Q_k$ is a set-theoretically independent collection of balls in the subspace H.*

Proof. All the atoms A_i $(i = 1, \ldots, 2^k)$ of the collection $Q_1, \ldots, Q_k$ are non-void, thus there exist points $p_i \in A_i$ $(i = 1, \ldots, 2^k)$. If $p_i \notin H$, then, by symmetry of the collection with respect to H, also the point p_i', symmetric to p_i with respect to H, belongs to A_i. By the connectedness of atoms, there exists an arc joining p_i with p_i', and completely contained in A_i. Obviously, this arc intersects H. Consequently, the atoms $H \cap A_i$ $(i = 1, \ldots, 2^k)$ are non-void.

Theorem 1. *A collection of balls $Q_1, \ldots, Q_k$ (where $k \leqslant n$) in the space E^n is set-theoretically independent if and only if the intersection of spheres of the balls $Q_1, \ldots, Q_k$ is an $(n-k)$-dimensional sphere.*

Proof. Denote by S_i the sphere of the ball Q_i ($i = 1, \ldots, k$).

(a) Let us suppose that the intersection $S = S_1 \cap \ldots \cap S_k$ is an $(n-k)$-dimensional sphere. The set S obviously contains at least two points, because $n-k \geqslant 0$. Let $\iota\colon M^n \to M^n$ be an inversion with centre at $s \in S$. The intersection $H_1 \cap \ldots \cap H_k$ of the hyperplanes $H_i = \iota(S_i \setminus \{s\})$ is an $(n-k)$-dimensional hyperplane, thus according to Lemma 4 there are 2^k components of $E^n \setminus (H_1 \cup \ldots \cup H_k)$. Consequently, by Lemma 6, there are 2^k non-void atoms of the collection $Q_1, \ldots, Q_k$, i. e., this collection is set-theoretically independent.

(b) The converse implication will be proved by induction. For $k = 1$ it trivially holds. Assume that it holds for $k-1$ and that a collection of balls $Q_1, \ldots, Q_k$ (where $k > 1$) is set-theoretically independent. Then the collection $Q_1, \ldots, Q_{k-1}$ is set-theoretically independent too. Hence the intersection $S' = S_1 \cap \ldots \cap S_{k-1}$ is an $(n-k+1)$-dimensional sphere. Let us suppose that the intersection

$$S_1 \cap \ldots \cap S_k = S' \cap S_k$$

is not an $(n-k)$-dimensional sphere. The sphere S_k being $(n-1)$-dimensional, our supposition is possible only in the following two cases:

$$\text{(i)} \quad S' \subset S_k, \qquad\qquad \text{(ii)} \quad S' \cap S_k \subset \{s'\}.$$

We shall eliminate both these possibilities. In the case (i) we have $S' = S_1 \cap \ldots \cap S_k$. Let us apply, just as in the part (a) of this proof, an inversion with centre at $s \in S'$. We obtain k $(n-1)$-dimensional hyperplanes, whose intersection is an $(n-k+1)$-dimensional hyperplane. Thus, according to Lemma 4, they cannot cut E^n into 2^k pieces. Hence, by Lemma 6, there are less than 2^k non-void atoms of the collection $Q_1, \ldots, Q_k$, and terefore this collection is not set-theoretically independent, contrary to the assumption. In the case (ii) the inequality $k \leqslant n$ implies that there exists a point s such that $s \in S'$ and $s \neq s'$. An inversion $\iota\colon M^n \to M^n$ with the centre at s maps the spheres $S_1, \ldots, S_{k-1}$ without the point s onto some $(n-1)$-dimensional hyperplanes $H_1, \ldots, H_{k-1}$, respectively. Let K denote the ball bounded by the sphere $\iota(S_k)$ in E^n. Since $k \leqslant n$, the intersection $H = H_1 \cap \ldots \cap H_{k-1}$ is at least a line. But (ii) implies $H \cap \iota(S_k) \subset \{\iota(s')\}$, and it follows that H lies outside K. In view of Lemma 3, the closed ball $\bar{K}$ does not intersect all the components of $E^n \setminus (H_1 \cup \ldots \cup H_{k-1})$. However, $\iota(\bar{K})$ contains Q_k or its complement. This shows, by Lemma 6, that the interior of an atom of the collection $Q_1, \ldots, Q_k$ is void, contrary to the assumption. The proof of Theorem 1 is complete.

Applying the above theorem to $k = n$ we see that the intersection of spheres in every set-theoretically independent collection of n balls in the space E^n is a two-point set.

THEOREM 2. *Let the collection of balls $Q_1, \ldots, Q_n$ in the space E^n be set-theoretically independent and let the two-point set $\{s, s'\}$ be the intersection of spheres of the balls $Q_1, \ldots, Q_n$. Let Q_{n+1} be a ball. The collection $Q_1, \ldots, Q_{n+1}$ is set-theoretically independent if and only if the ball Q_{n+1} contains one of the points s and s', and its closure $\bar{Q}_{n+1}$ does not contain the remaining one.*

Proof. Denote by S_i the sphere of the ball Q_i $(i = 1, \ldots, n+1)$.

(a) We can assume that $s \notin \bar{Q}_{n+1}$ and $s' \in Q_{n+1}$. After an inversion ι: $M^n \to M^n$ with the centre at s we get the $(n-1)$-dimensional hyperplanes $H_i = \iota(S_i \setminus \{s\})$ which intersect at a single point $p = \iota(s')$ and the ball $\iota(Q_{n+1})$ which contains p. By Lemma 5, this ball contains points from all the components of $E^n \setminus (H_1 \cup \ldots \cup H_n)$. Obviously, all those components, being unbounded by Lemma 1, have common points also with the complement of Q_{n+1}. Thus, by Lemma 6, the collection of balls $Q_1, \ldots, Q_{n+1}$ is set-theoretically independent.

(b) We shall prove that the collection $Q_1, \ldots, Q_{n+1}$ is not set-theoretically independent in each of the following four cases:

(i)	$s \notin \bar{Q}_{n+1},\ s' \notin \bar{Q}_{n+1},$	(ii)	$s \in S_{n+1},\ s' \notin S_{n+1},$
(iii)	$s \in S_{n+1},\ s' \in S_{n+1},$	(iv)	$s \in Q_{n+1},\ s' \in Q_{n+1}.$

In the cases (i), (ii), and (iii), we do the same as in the part (a) above. Consequently, in the case (i) we get the ball $\iota(Q_{n+1})$, disjoint with $H_1 \cap \ldots \cap H_n = \{p\}$. By Lemma 3, this ball does not meet all the components of $E^n \setminus (H_1 \cup \ldots \cup H_n)$. It follows from Lemma 6 that Q_{n+1} does not meet the interior of an atom of the collection $Q_1, \ldots, Q_n$. In the case (ii) we get the $(n-1)$-dimensional hyperplane $H_{n+1} = \iota(S_{n+1} \setminus \{s\})$, which does not pass through p. By Lemma 3, the hyperplane H_{n+1} does not meet any component B of $E^n \setminus (H_1 \cup \ldots \cup H_n)$. Therefore the closure $\bar{P}$ of one of the half-spaces on which H_{n+1} cuts E^n does not meet B. However, the set $\iota(\bar{P})$ contains Q_{n+1} or its complement. It thus follows from Lemma 6 that Q_{n+1} or its complement does not meet the interior of an atom of the collection $Q_1, \ldots, Q_n$. In the case (iii) we get the hyperplane H_{n+1} passing through p. In view of Lemma 2, both half-spaces on which H_{n+1} cuts E^n contain the whole of some components of $E^n \setminus (H_1 \cup \ldots \cup H_n)$. Since one of these half-spaces is the set $\iota(Q_{n+1})$ and $s \in S_{n+1} \subset \bar{Q}_{n+1}$, it follows from Lemma 6 that the closure $\bar{Q}_{n+1}$ contains an atom of the collection $Q_1, \ldots, Q_n$. The case (iv) reduces to the case (iii), for we can find a ball contained in Q_{n+1} and such that both points s and s' lie on its sphere.

THEOREM 3. *If a collection of balls $Q_1, \ldots, Q_k$ (where $k \leqslant n+1$) in the space E^n is set-theoretically independent, then the intersection of spheres of the balls $Q_1, \ldots, Q_k$ is the set of all points $x \in E^n$ such that x is an accumulation point of each atom of the collection $Q_1, \ldots, Q_k$.*

Proof. Let $x \in S = S_1 \cap \ldots \cap S_k$, where S_i is the sphere of the ball Q_i ($i = 1, \ldots, k$). Therefore $S \neq \emptyset$ and Theorem 2 implies $k \leqslant n$. Take a ball Q_{k+1} with centre x and with a radius sufficiently small, so that the sphere of Q_{k+1} cuts E^n between some points of S. According to Theorems 1 and 2, the collection $Q_1, \ldots, Q_{k+1}$ is set-theoretically independent, and thus Q_{k+1} has points from all the atoms of the collection $Q_1, \ldots, Q_k$.

Conversely, if $y \notin S$, then $y \notin S_i$ for some $i = 1, \ldots, k$. Therefore a ball with centre y and with a radius sufficiently small either does not intersect Q_i or its complement.

THEOREM 4. *Every set-theoretically independent collection of balls in the space E^n can be extended to a maximal set-theoretically independent collection, which consists of $n+1$ balls.*

Proof. It is known (2) that every set-theoretically independent collection of balls in the space E^n consists at most of $n+1$ balls. Let a collection of balls $Q_1, \ldots, Q_k$ (where $k \leqslant n$) be set-theoretically independent in E^n. By Theorem 1, the intersection of spheres of the balls $Q_1, \ldots, Q_k$ is an $(n-k)$-dimensional sphere, thus it contains at least two points. Construct a ball Q_{k+1} with centre at one of them and such that its closure $\bar{Q}_{n+1}$ does not contain the remaining one. According to Theorem 3, the collection $Q_1, \ldots, Q_{k+1}$ is set-theoretically independent.

THEOREM 5. *All atoms of a set-theoretically independent collection of balls in the space E^n (where $n > 1$) are connected* (3).

Proof. Let $Q_1, \ldots, Q_k$ be a set-theoretically independent collection of balls in E^n. Then, as we know (2), we must have $k \leqslant n+1$. Denote by S_i the sphere of the ball Q_i ($i = 1, \ldots, k$).

If $k \leqslant n$, then by Theorem 1 the intersection $S = S_1 \cap \ldots \cap S_k$ is an $(n-k)$-dimensional sphere. Let $\iota: M^n \to M^n$ be an inversion with

(2) A. Rényi, C. Rényi et J. Surányi, *Sur l'indépendance des domaines simples dans l'espace euclidien à n dimensions*, Colloquium Mathematicum 2 (1951), p. 130-135.

(3) A. Lelek has generalized Theorem 5 to the following form:

If an atom of a finite collection of open or closed balls in the space E^n (where $n > 1$) is non-void, then the opposite atom of the collection is connected.

Here by the *opposite* atom to an atom corresponding to a sequence $j_1, \ldots, j_k$ one means the atom corresponding to the sequence $-j_1, \ldots, -j_k$.

a centre $s \in S$. According to Lemma 6, each atom A of the collection $Q_1, \ldots, Q_k$ corresponds to a component B of $E^n \setminus (H_1 \cup \ldots \cup H_k)$, where $H_i = \iota(S_i \setminus \{s\})$. The component B is unbounded, by Lemma 1, and therefore s is an accumulation point of the set $\iota(B \setminus \{s\})$. Hence, the inclusions from Lemma 6 give

$$\iota(B \setminus \{s\}) \subset A \subset \overline{\iota(B \setminus \{s\})},$$

which implies the connectedness of the atom A. Indeed, as $n > 1$, no point cuts B that is an open connected subset of the space E^n.

If $k = n+1$, then according to Theorems 1 and 2, the intersection $S_1 \cap \ldots \cap S_n$ is a two-point set $\{s, s'\}$ such that s' belongs to Q_{n+1} and s does not belong to its closure $\overline{Q}_{n+1}$. Applying the same inversion ι we get the ball $\iota(Q_{n+1})$ which contains the origin of the oblique coordinate system constructed in Lemma 1. It follows that both sets

$$B' = (B \setminus \{s\}) \cap \iota(Q_{n+1}), \qquad B'' = (B \setminus \{s\}) \setminus \iota(Q_{n+1})$$

are connected and satisfy the inclusions

$$\iota(B') \subset A \cap Q_{n+1} \subset \overline{\iota(B')}, \qquad \iota(B'') \subset A \setminus Q_{n+1} \subset \overline{\iota(B'')},$$

which yield the connectedness of all atoms of the collection $Q_1, \ldots, Q_{n+1}$.

Remark. In the proof of Theorem 5, in the case $k \leqslant n$, the set-theoretical independence of the collection of balls $Q_1, \ldots, Q_k$ was used only to assure the existence of a point belonging to all the spheres $S_1, \ldots, S_k$ and the unboundedness of each component of $E^n \setminus (H_1 \cup \ldots \cup H_k)$. The last condition is always fulfilled provided that $k \leqslant n$. Thus for $k \leqslant n$ we could have a stronger theorem, namely that if the intersection of spheres of the balls $Q_1, \ldots, Q_k$ in the space E^n is non-void, then all atoms of the collection $Q_1, \ldots, Q_k$ are connected. The condition $k \leqslant n$ is necessary here, as is shown by an example even for the plane E^2 (see fig. 1).

THEOREM 6. *The centres of balls that form a set-theoretically independent collection in the space E^n constitute a linearly independent system of points in E^n* [4].

Proof. The statement is obvious for $k \leqslant 2$. Let a collection of balls $Q_1, \ldots, Q_k$ (where $k > 2$) be set-theoretically independent in E^n. Then, as we have seen in Theorem 4, the inequality $k \leqslant n+1$ holds. If the centres of the balls $Q_1, \ldots, Q_k$ were not linearly independent, they would lie all on a $(k-2)$-dimensional hyperplane H. In view of Theorem 5

[4] Theorems 6 and 7 give an answer to an unpublished problem raised by B. Węglorz.

and the inequality $k-1 > 1$, we could apply Lemma 7 successively $n-k+2$ times. The resulting collection of $(k-2)$-dimensional balls would be set-theoretically independent in the $(k-2)$-dimensional subspace H and would consist of k balls. But this is impossible, since each such collection can consist at most of $(k-2)+1 = k-1$ balls.

THEOREM 7. *If a system of points $p_1, \ldots, p_k$ in the space E^n (where $1 < k \leqslant n+1$) is linearly independent, then the balls with centres at the points $p_1, \ldots, p_k$, respectively, and with equal radii, greater than the radius of the $(k-1)$-dimensional ball circumscribed on the points $p_1, \ldots, p_k$* (5)*, form a set-theoretically independent collection of balls in E^n.*

Proof. Let us notice that the case $k < n+1$ reduces to the case $k = n+1$. Indeed, cutting the balls by the $(k-1)$-dimensional hyperplane H^{k-1}, that passes through the points $p_1, \ldots, p_k$, we obtain the $(k-1)$-dimensional balls in the subspace H^{k-1} which have centres at the points $p_1, \ldots, p_k$ and equal radii. The set-theoretical independence of the so obtained collection of balls in H^{k-1} implies that of the original collection in E^n.

Now, we shall prove the theorem for $k = n+1$, by induction on n. If $n = 1$, the theorem holds, because every two one-dimensional balls on the line E^1, i. e., two intervals, with centres at the points p_1 and p_2, respectively, and of the same length, greater than the distance between p_1 and p_2, must intersect, thus they must form a set-theoretically independent collection in E^1. Assume the theorem holds for n points and consider a system of points $p_1, \ldots, p_{n+1}$ linearly independent in E^n. The radius ϱ_n of the n-dimensional ball C^n circumscribed on the points $p_1, \ldots, p_{n+1}$ is not less than the radius ϱ_{n-1} of the $(n-1)$-dimensional ball C^{n-1} circumscribed on the points $p_1, \ldots, p_n$. It follows that the balls $Q_1, \ldots, Q_n$ with centres at the points $p_1, \ldots, p_n$, respectively, and with radii equal to $\varrho > \varrho_n \geqslant \varrho_{n-1}$ form a set-theoretically independent collection in E^n. By Theorem 1, the intersection of spheres of the balls $Q_1, \ldots, Q_n$ is a two-point set $\{s, s'\}$.

Let H be the $(n-1)$-dimensional hyperplane passing through $p_1, \ldots, p_n$. Since the system $p_1, \ldots, p_{n+1}$ is linearly independent, we have $p_{n+1} \in E^n \setminus H$.

The points s and s' have the same distance ϱ from all the points $p_1, \ldots, p_n$. Thus s and s' lie symmetrically with respect to H, on the line perpendicular to H and going through the centre of the ball C^{n-1}. We have also $C^{n-1} = H \cap C^n$. Moreover, the equalities

$$C^{n-1} = H \cap K = H \cap K'$$

(5) That is a ball whose boundary is the $(k-1)$-dimensional sphere circumscribed on the points $p_1, \ldots, p_k$ in the $(k-1)$-dimensional hyperplane generated by them.

hold, where K and K' are $(n-1)$-dimensional balls with the same radii equal to ϱ and with centres s and s', respectively. So we see that all three balls C^n, K and K' belong to the bundle of balls in E^n, determined by the $(n-1)$-dimensional ball C^{n-1}. Let P denote the open half-space — one of the components of $E^n \setminus H$ — to which p_{n+1} belongs. Since $\varrho_n < \varrho$ and only the balls K, K' in this bundle have radius ϱ, the cup $P \cap C^n$ is contained either in the ball K or in the ball K'. We can assume that

$$P \cap \overline{C^n} \subset K,$$

but then we also have

$$P \cap \overline{K'} \subset C^n$$

(fig. 2). It follows from the two last inclusions that $p_{n+1} \in K \setminus \overline{K'}$. In fact, p_{n+1} lies on the sphere of the ball C^n and it belongs to the half-space P.

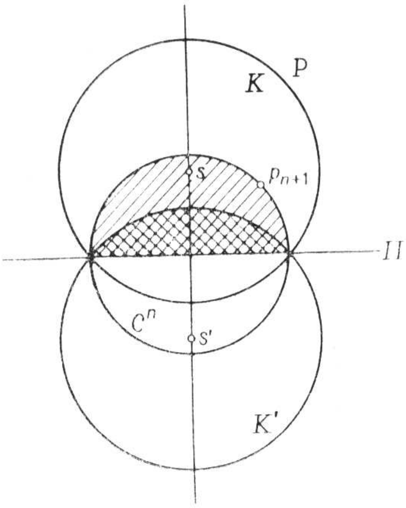

Fig. 2

Therefore $p_{n+1} \in K$ and $p_{n+1} \notin \overline{K}'$. Hence the distance between p_{n+1} and s is less than ϱ while the distance between p_{n+1} and s' is greater than ϱ. Thus the ball Q_{n+1}, having centre p_{n+1} and radius ϱ, contains s and its closure $\overline{Q}_{n+1}$ does not contain s'. According to Theorem 2, the collection of the balls $Q_1, \ldots, Q_{n+1}$ is set-theoretically independent.

Reçu par la Rédaction le 30. 6. 1964