ON ALGEBRAIC OPERATIONS IN IDEMPOTENT ALGEBRAS

BY

K. URBANIK (WROCLAW)

I. Introduction. In this paper we adopt the definitions and notation given by Marczewski in [2] and [3]. Let \( \mathcal{A} = (A; F) \) be an algebra, i.e. a set \( A \) of elements and a class \( F \) of fundamental operations consisting of \( A \)-valued functions of several variables running over \( A \). If \( A = \{a_1, a_2, \ldots\} \) and \( F = \{f, g, \ldots\} \), we shall sometimes write \( (a_1, a_2, \ldots; f, g, \ldots) \) or \( (A; f, g, \ldots) \) instead of \( (A; F) \). The \( n \)-ary operations

\[
e_k^n(x_1, x_2, \ldots, x_n) = x_k \quad (k = 1, 2, \ldots, n; \quad n = 1, 2, \ldots)
\]

will be called trivial. We denote by \( A \) the class of all algebraic operations, i.e. the smallest class containing trivial operations and closed under the composition with fundamental operations. The subclass of all \( n \)-ary algebraic operations will be denoted by \( A^{(n)} \). If \( 1 \leq k \leq n \), then \( A^{(n,k)} \) will denote the subclass of \( A^{(n)} \) consisting of all operations depending on at most \( k \) variables. Thus \( f \in A^{(n,k)} \) if there is an operation \( g \in A^{(k)} \) such that \( f(x_1, x_2, \ldots, x_n) = g(x_{i_1}, x_{i_2}, \ldots, x_{i_k}) \) for a system of indices \( i_1, i_2, \ldots, i_k \). Two algebras \( (A; F_1) \) and \( (A; F_2) \) having the same class of algebraic operations will be treated here as identical. In particular, we have the equation \( (A; F) = (A; A) \).

The algebra \( (A; F) \) is called idempotent if \( A^{(1)} \) consists of trivial operations only. In other words, the algebra is idempotent if and only if for every algebraic operation \( f \) the equation \( f(x, x, \ldots, x) = x \) holds.

Let \( \mathcal{S}(\mathcal{A}) \) be the set of all non-negative integers \( n \) for which there exists an algebraic non-trivial \( n \)-ary operation in \( \mathcal{A} \) depending on every variable. The investigation of the sets \( \mathcal{S}(\mathcal{A}) \) was suggested by Marczewski. In particular, he proved in [5] that if there is no constant operation in the algebra \( \mathcal{A} \) and there is an \( n \)-ary symmetrical (or even quasi-symmetrical) operation, then the set \( \mathcal{S}(\mathcal{A}) \) contains the arithmetical progression \( n + (n-1)k \) \( (k = 0, 1, \ldots) \). This result for \( n = 2 \) was previously obtained by Plonka in [7]. The aim of the present paper is to give a complete description of all possible sets \( \mathcal{S}(\mathcal{A}) \).
II. Examples. First of all we shall prove that for any subset $E$ of the set $\{0, 1, \ldots\}$ satisfying the condition $E \cap \{0, 1\} \neq \emptyset$ there exists an algebra $\mathfrak{A}_E$ for which $\mathcal{S}(\mathfrak{A}_E) = E$.

Let $A$ be the set of all non-negative integers. We define an $n$-ary operation $t_n (n \geq 1)$ on $A$ as follows: $t_n(x_1, x_2, \ldots, x_n) = 2$ if the integers $x_1, x_2, \ldots, x_n$ are all different and odd and $t_n(x_1, x_2, \ldots, x_n) = 0$ in the opposite case. Moreover, we define a constant operation $t_0(x) = 0$ for all $x \in A$. Of course, the operations $t_n$ depend on every variable.

Suppose that $0 \in E$. Put $F_E = \{t_n : n \in E\}$ and $\mathfrak{A}_E = (A; F_E)$. From the definition of fundamental operations $t_n$ it follows that each non-trivial algebraic operation in $\mathfrak{A}_E$ is of the form

$$f(x_1, x_2, \ldots, x_k) = t_n(x_{j_1}, x_{j_2}, \ldots, x_{j_n}),$$

where $n \in E$, $1 \leq j_i \leq k \ (i = 1, 2, \ldots, n)$ and all indices $j_1, j_2, \ldots, j_n$ are different. Thus the equation $\mathcal{S}(\mathfrak{A}_E) = E$ is true.

Now for any positive integer $n$ we define an $n$-ary operation $w_n$ as follows: $w_n(x_1, x_2, \ldots, x_n) = 2x_n$ if the integers $x_1, x_2, \ldots, x_n$ are different and odd and $w_n(x_1, x_2, \ldots, x_n) = 2x_1$ in the opposite case. Of course, each operation $w_n$ is non-trivial and depends on every variable.

Suppose that $0 \in E$ and $1 \in E$. Put $F_E = \{w_n : n \in E\}$ and $\mathfrak{A}_E = (A; F_E)$. It is very easy to prove that each algebraic non-trivial operation in the algebra $\mathfrak{A}_E$ is of the form

$$f(x_1, x_2, \ldots, x_k) = 2^r w_n(x_{j_1}, x_{j_2}, \ldots, x_{j_n}),$$

where $n \in E$, $r \geq 0$, $1 \leq j_i \leq k \ (i = 1, 2, \ldots, n)$ and the indices $j_1, j_2, \ldots, j_n$ are all different. Hence the equation $\mathcal{S}(\mathfrak{A}_E) = E$ follows, which completes the proof.

Consequently, it remains the question of a characterization of all subsets $E \subset \{2, 3, \ldots\}$ for which there exists an algebra $\mathfrak{A}$ with $\mathcal{S}(\mathfrak{A}) = E$. Obviously, this question is simply the question of a description of all possible sets $\mathcal{S}(\mathfrak{A})$ for idempotent algebras $\mathfrak{A}$.

Now we shall give some examples of the sets $\mathcal{S}(\mathfrak{A})$ for idempotent algebras $\mathfrak{A}$.

1. Let $\mathfrak{A}$ be a trivial algebra, i.e. an algebra in which all algebraic operations are trivial. In this case the set $\mathcal{S}(\mathfrak{A})$ is empty. Of course, this property characterizes the trivial algebras among idempotent ones.

2. Let $G$ be a Boolean group with the addition as a group operation, i.e. a group in which all elements different from the zero element are of order 2. Put $g(x, y, z) = x + y + z$ for $x, y, z \in G$ and $\mathfrak{A} = (G; g)$. It is easy to see that each algebraic operation in $\mathfrak{A}$ is a sum of an odd number of different variables. If $G$ contains at least two elements, then each
operation \( x_1 + x_2 + \ldots + x_{2k+1} \) depends on every variable. Consequently, in this case the set \( \mathcal{F}(\mathfrak{A}) \) consists of all odd integers greater than 1.

3. Consider a diagonal algebra, i.e., an algebra of the form \( \mathfrak{A} = (A; d) \), where the set \( A \) is a Cartesian product \( A_1 \times A_2 \times \ldots \times A_n \) and the fundamental \( n \)-ary operation \( d \) is defined by the formula

\[
d(\langle a_1^1, a_2^1, \ldots, a_n^1 \rangle, \langle a_1^2, a_2^2, \ldots, a_n^2 \rangle, \ldots, \langle a_1^n, a_2^n, \ldots, a_n^n \rangle) = \langle a_1^1, a_2^2, \ldots, a_n^n \rangle,
\]

where \( a_j^i \in A_j \quad (i, j = 1, 2, \ldots, n) \). Diagonal algebras were introduced by Plonka in [7]. If the \( n \)-ary operation \( d \) depends on every variable, then the algebra \( \mathfrak{A} \) is called \( n \)-dimensional. Obviously, one-dimensional diagonal algebras are trivial. If \( \mathfrak{A} \) is an \( n \)-dimensional diagonal algebra and \( n \geq 2 \), then, according to [7], the set \( \mathcal{F}(\mathfrak{A}) \) consists of the integers \( 2, 3, \ldots, n \).

We note that diagonal algebras can be defined in terms of binary fundamental operations. Namely, we have the equation \( (A; d) = (A; d_1, d_2, \ldots, d_n) \), where

\[
d_j(\langle a_1, a_2, \ldots, a_n \rangle, \langle b_1, b_2, \ldots, b_n \rangle) = \langle b_1, b_2, \ldots, b_{j-1}, a_j, b_{j+1}, \ldots, b_n \rangle \quad (j = 1, 2, \ldots, n)
\]

for \( a_k, b_k \in A_k \quad (k = 1, 2, \ldots, n) \).

4. Let \( A \) be an arbitrary set containing at least \( m \) elements, where \( m \geq 3 \). Let \( l_m \) be an \( m \)-ary operation on \( A \) defined as follows: \( l_m(x_1, \ldots, x_m) = x_i \) if \( x_1, \ldots, x_m \) are all different and \( l_m(x_1, \ldots, x_m) = x_m \) in the opposite case (see [4], p. 2). Put \( \mathfrak{A} = (A; l_m) \). Taking into account that the operation \( l_m \) depends on every variable and applying Lemma 1 proved in Chapter IV, we infer that the set \( \mathcal{F}(\mathfrak{A}) \) consists of all integers \( \geq m \).

Further, let \( A \) be a three-element set and let the binary operation \( r_3 \) on \( A \) be defined by the conditions: if \( x \neq y \), then \( r_3(x, y) \notin \{x, y\} \) and \( r_3(x, x) = x \) (see [4], p. 2). The operation \( r_3 \) is symmetrical. Thus, by previously cited Plonka's result [7], the set \( \mathcal{F}(\mathfrak{A}) \) for the algebra \( \mathfrak{A} = (A; r_3) \) contains all integers \( \geq 2 \).

5. Let \( G \) be an infinite Boolean group and \( (G; g) \) the algebra defined in Example 2. Given an integer \( m \geq 5 \) we define an \( m \)-ary operation \( p_m \) on \( G \) as follows: \( p_m(x_1, x_2, \ldots, x_m) = x_m \) if all elements \( x_1, x_2, \ldots, x_m \) belong to a subalgebra of the algebra \( (G; g) \) generated by less than \( m \) elements and \( p_m(x_1, x_2, \ldots, x_m) = x_1 \) in the opposite case. Put \( \mathfrak{A} = (G; g, p_m) \). It is very easy to verify that for \( k < m \) every \( k \)-ary algebraic operation in \( \mathfrak{A} \) is also algebraic in \( (G; g) \). Thus the intersection \( \mathcal{F}(\mathfrak{A}) \cap \{2, 3, \ldots, m-1\} \) consists of all odd integers less than \( m \). Hence and from the inclusion \( \mathcal{F}(\mathfrak{A}) \supset \mathcal{F}((G; g)) \cup \mathcal{F}((G; p_m)) \) it follows, by
Lemma 1, that the set $\mathcal{I}(\mathfrak{A})$ consists of all integers $\geq m$ and all odd integers greater than 1.

6. Consider an infinite $n$-dimensional diagonal algebra $(A; d)$, where $n \geq 2$. Given an integer $m > n$ we define an $m$-ary operation $q_m$ on $A$ by the conditions: $q_m(x_1, x_2, \ldots, x_m) = x_m$ if the elements $x_1, x_2, \ldots, x_m$ belong to a subalgebra of the diagonal algebra $(A; d)$ generated by less than $m$ elements and $q_m(x_1, x_2, \ldots, x_m) = x_1$ in the opposite case. Put $\mathfrak{A} = (A; d, q_m)$. Since the set $A$ is infinite, the operation $q_m$ depends on every variable. Moreover, $\mathcal{I}(\mathfrak{A}) \cap \{2, 3, \ldots, m-1\} = \{2, 3, \ldots, n\}$. Hence and from the inclusion $\mathcal{I}(\mathfrak{A}) \supset \mathcal{I}(\langle A; d \rangle) \cup \mathcal{I}(\langle A; q_m \rangle)$ it follows, by Lemma 1, that the set $\mathcal{I}(\mathfrak{A})$ consists of all integers $s$ satisfying one of the inequalities $2 \leq s \leq n$, $s \geq m$.

III. Theorems. The examples give six types of the sets $\mathcal{I}(\mathfrak{A})$ for idempotent algebras $\mathfrak{A}$. The main result of the present paper is that these six types give a complete description of all possible sets $\mathcal{I}(\mathfrak{A})$ for idempotent algebras $\mathfrak{A}$.

**Theorem 1.** For each idempotent algebra $\mathfrak{A}$ one of the following cases holds:

(i) $\mathcal{I}(\mathfrak{A})$ is an empty set,
(ii) $\mathcal{I}(\mathfrak{A})$ consists of all odd integers greater than 1,
(iii) $\mathcal{I}(\mathfrak{A})$ consists of all integers $s$ satisfying the inequality $2 \leq s \leq n$, where $n \geq 2$,
(iv) $\mathcal{I}(\mathfrak{A})$ consists of all integers $\geq m$, where $m \geq 2$,
(v) $\mathcal{I}(\mathfrak{A})$ consists of all odd integers greater than 1 and all integers $\geq m$, where $m \geq 5$.
(vi) $\mathcal{I}(\mathfrak{A})$ consists of all integers $s$ satisfying one of the inequalities $2 \leq s \leq n, s \geq m$, where $m > n \geq 2$.

Sometimes the set $\mathcal{I}(\mathfrak{A})$ completely determines the algebraic structure of an idempotent algebra $\mathfrak{A}$. For instance, the set $\mathcal{I}(\mathfrak{A})$ for an idempotent algebra is empty if and only if the algebra is trivial. We shall prove that except two cases $\mathcal{I}(\mathfrak{A}) = \{2, 3, \ldots\}$ and $\mathcal{I}(\mathfrak{A}) = \{3, 4, \ldots\}$ the set $\mathcal{I}(\mathfrak{A})$ determines the algebraic structure of the idempotent algebra $\mathfrak{A}$.

**Theorem 2.** Let $\mathfrak{A}$ be an idempotent algebra.

1. $\mathcal{I}(\mathfrak{A}) = \{s: 2 \leq s \leq n\}$, where $n \geq 1$ if and only if $\mathfrak{A}$ is a diagonal algebra.

2. $\mathcal{I}(\mathfrak{A}) = \{s: 2 \leq s \leq n\} \cup \{s: s \geq m\}$, where $m > n \geq 1$, $m \geq 4$ if and only if $\mathfrak{A} = (A; \{d\} \cup \mathbf{F})$, where $(A; d)$ is an $n$-dimensional diagonal algebra, the class $\mathbf{F}$ contains an $m$-ary operation depending on every
variable, all operations \( f \) from \( F \) depend on at least \( m \) variables and satisfy the equation
\[
f(x_1, x_2, \ldots, x_k) = x_1
\]
whenever the elements \( x_1, x_2, \ldots, x_k \) belong to a subalgebra of the algebra \( (A; d) \) generated by less than \( m \) elements.

3. \( \mathcal{F}(\mathcal{U}) \) consists of all odd integers greater than 1 if and only if \( \mathcal{U} = (G; g) \), where \( G \) is an at least two-element Boolean group and \( g(x, y, z) = x + y + z \).

4. \( \mathcal{F}(\mathcal{U}) \) consists of all odd integers greater than 1 and all integers \( \geq m \), where \( m \geq 5 \) if and only if \( \mathcal{U} = (G; \{g\} \circ F) \), where \( (G; g) \) is the algebra defined in the preceding assertion, \( F \) contains an \( m \)-ary operation depending on every variable, all operations \( f \) from \( F \) depend on at least \( m \) variables and satisfy the equation \( f(x_1, x_2, \ldots, x_k) = x_1 \) whenever the elements \( x_1, x_2, \ldots, x_k \) belong to a subalgebra of the algebra \( (G; g) \) generated by less than \( m \) elements.

As a simple consequence of Theorems 1 and 2 we obtain the following results which are a generalization of a characterization theorem for diagonal algebras presented in [7].

**Theorem 3.** Let \( \mathcal{U} \) be an idempotent algebra. The set \( \mathcal{F}(\mathcal{U}) \) is finite if and only if \( \mathcal{U} \) is a diagonal algebra.

**Theorem 4.** Let \( \mathcal{U} \) be an idempotent algebra. The set \( \mathcal{F}(\mathcal{U}) \) and its complement are both infinite if and only if \( \mathcal{U} = (G; g) \), where \( G \) is an at least two-element Boolean group and \( g(x, y, z) = x + y + z \).

**Theorem 5.** Let \( \mathcal{U} \) be an idempotent algebra. If \( n \notin \mathcal{F}(\mathcal{U}) \) and all fundamental operations in \( \mathcal{U} \) depend on at most \( n-1 \) variables, then either \( \mathcal{U} \) is a diagonal algebra or \( \mathcal{U} = (G; g) \), where \( G \) is a Boolean group and \( g(x, y, z) = x + y + z \).

**Theorem 6.** Let \( \mathcal{U} \) be an idempotent algebra with binary fundamental operations. If \( \mathcal{F}(\mathcal{U}) \neq \{2, 3, \ldots\} \), then \( \mathcal{U} \) is a diagonal algebra.

The proof of Theorems 1 and 2 will be given in the next section. Before proving the Theorems we shall prove some Lemmas.

**IV. Lemmas and proof of the Theorems.** In this section we assume that considered algebras are idempotent.

**Lemma 1.** Let \( m \geq 3 \) and let \( \mathcal{U} = (A; f) \), where \( f \) is an \( m \)-ary operation in \( A \) satisfying the equation
\[
f(x_1, x_2, \ldots, x_m) = x_1
\]
whenever at least two elements among \( x_1, x_2, \ldots, x_m \) are equal. If the operation \( f \) depends on every variable, then \( \mathcal{F}(\mathcal{U}) = \{s : s \geq m\} \).

**Proof.** From the assumption (2) it follows that the composition \( f(e_1^{(m-1)}, e_2^{(m-1)}, \ldots, e_{jm}^{(m-1)}) \) \( (1 \leq j_k \leq m-1; k = 1, 2, \ldots, m) \) is the trivial
operation $e_{1}^{(m-1)}$. Thus the class of $(m-1)$-ary trivial operations is closed under the composition with the fundamental operation $f$. Hence it follows that $A^{(m-1)}$ consists of trivial operations. Consequently, to prove the Lemma it suffices to prove that for each integer $k \geq m$ there exists an algebraic $k$-ary operation $f_{k}$ depending on every variable. We shall prove slightly stronger statement by induction with respect to $k$. Namely, we shall prove that the operation $f_{k}$ satisfies the additional condition

(3) \quad f_{k}(x, y, y, \ldots, y) = x.

Set $f_{m} = f$. By assumption (2) the operation $f_{m}$ satisfies equation (3) and depends on every variable. Given an algebraic $k$-ary operation $f_{k}$ ($k \geq m$) depending on every variable and satisfying equation (3), we put

(4) \quad f_{k+1}(x_{1}, x_{2}, \ldots, x_{k+1}) = f(f_{k}(x_{1}, x_{2}, \ldots, x_{k}), x_{k+1}, x_{2}, x_{3}, \ldots, x_{m-1}).

Of course, the operation $f_{k+1}$ is algebraic and the equation

\[ f_{k+1}(x, y, y, \ldots, y) = f(f_{k}(x, y, y, \ldots, y), y, y, \ldots, y) = x \]

holds. It remains to prove that $f_{k+1}$ depends on every variable. Taking into account the inequality $m-1 \geq 2$ and setting $x_{k+1} = x_{2}$ into (4) we get, in view of (2), the equation

\[ f_{k+1}(x_{1}, x_{2}, \ldots, x_{k}, x_{2}) = f_{k}(x_{1}, x_{2}, \ldots, x_{k}), \]

which shows that the operation $f_{k+1}$ depends on the variables $x_{1}, x_{3}, x_{4}, \ldots, x_{k}$.

Since all $(m-1)$-ary algebraic operations are trivial, the operation $f_{k}(x_{1}, x_{2}, \ldots, x_{m-2}, x_{k}, x_{k}, \ldots, x_{k})$ is trivial too. Thus, by (3), we have the equation

(5) \quad f_{k}(x_{1}, x_{2}, \ldots, x_{m-2}, x_{k}, x_{k}, \ldots, x_{k}) = x_{1}.

Setting $x_{j} = x_{k}$ ($j = m-1, m, \ldots, k$) into (4) we get, in view of (5), the equation

\[ f_{k+1}(x_{1}, x_{2}, \ldots, x_{m-2}, x_{k}, x_{k}, \ldots, x_{k}, x_{k+1}) = f(x_{1}, x_{k+1}, x_{2}, x_{3}, \ldots, x_{m-2}, x_{k}), \]

which, in view of the inequality $k \geq m$, shows that the operation $f_{k+1}$ depends on the variables $x_{2}$ and $x_{k+1}$. Consequently, the operation $f_{k+1}$ depends on every variable, which completes the proof of the Lemma.

**Lemma 2.** Suppose that there exists a ternary algebraic operation $f$ in the algebra $A$ depending on every variable and satisfying the condition

(6) \quad f(x, y, y) = x.
If \( s \in \mathcal{F}(\mathcal{A}) \), then \( s + 2 \in \mathcal{F}(\mathcal{A}) \).

Proof. Suppose that \( s \in \mathcal{F}(\mathcal{A}) \). Let \( g \) be an \( s \)-ary algebraic operation depending on every variable. Put

\[
h(x_1, x_2, \ldots, x_{s+2}) = f(g(x_1, x_2, \ldots, x_s), x_{s+1}, x_{s+2}).
\]

Since

\[
h(x, x, \ldots, x, x_{s+1}, x_{s+2}) = f(g(x, x, \ldots, x), x_{s+1}, x_{s+2}) = f(x, x_{s+1}, x_{s+2})
\]

and, by (6),

\[
h(x_1, x_2, \ldots, x_s, y, y) = f(g(x_1, x_2, \ldots, x_s), y, y) = g(x_1, x_2, \ldots, x_s),
\]

we infer that the \((s + 2)\)-ary algebraic operation \( h \) depends on every variable. Thus, \( s + 2 \in \mathcal{F}(\mathcal{A}) \).

Lemma 3. Suppose that there exists a ternary algebraic operation \( g \) in the algebra \( \mathcal{A} \) depending on every variable and satisfying the condition

\[
g(x, y, y) = y.
\]

If \( 2 \not\in \mathcal{F}(\mathcal{A}) \), then \( \mathcal{F}(\mathcal{A}) = \{s : s \geq 3\} \).

Proof. Since the only binary algebraic operations in \( \mathcal{A} \) are trivial ones, we have one of the following cases:

\[
g(y, x, y) = y, \quad g(y, y, x) = x, \quad (8) \\
g(y, x, y) = x, \quad g(y, y, x) = y, \quad (9) \\
g(y, x, y) = y, \quad g(y, y, x) = y, \quad (10) \\
g(y, x, y) = x, \quad g(y, y, x) = x. \quad (11)
\]

Setting

\[
f(x, y, z) = \begin{cases} 
  g(z, x, y) & \text{in the case (8)}, \\
  g(y, x, z) & \text{in the case (9)},
\end{cases}
\]

we get a ternary algebraic operation depending on every variable and satisfying the equation \( f(x, y, z) = x \) whenever at least two elements among \( x, y, z \) are equal. Hence, by Lemma 1, we get the assertion of the Lemma in both cases (8) and (9).

Put

\[
p^*(x, y, z) = \begin{cases} 
  g(x, y, z) & \text{in the case (10)}, \\
  g(g(x, y, z), y, z) & \text{in the case (11)}.
\end{cases}
\]

It is very easy to verify that the operation \( p^* \) satisfies the equations

\[
p^*(x, x, y) = p^*(x, y, x) = p^*(y, x, x) = x. \quad (12)
\]
The algebra in question is non-trivial. Consequently, it contains at least two elements. Denoting by 0 and 1 a pair of elements of the algebra \( A \) we infer, in virtue of (12), that the set \( \{0, 1\} \) is closed under the operation \( p^* \). Moreover, the algebra \( (0, 1; p^*) \) is the Post algebra \( \mathcal{P}^* \) (see [6], p. 200). Thus, \( \mathcal{P}(\mathcal{P}^*) \subset \mathcal{P}(\mathfrak{A}) \). But \( \mathcal{P}(\mathcal{P}^*) = \{s : s \geq 3\} \) (see [6], p. 202), which completes the proof of the Lemma in the cases (10) and (11).

**Lemma 4.** If \( 2 \in \mathcal{P}(\mathfrak{A}) \), \( 3 \in \mathcal{P}(\mathfrak{A}) \) and \( 4 \in \mathcal{P}(\mathfrak{A}) \), then \( \mathcal{P}(\mathfrak{A}) = \{s : s \geq 3\} \).

**Proof.** Since \( 3 \in \mathcal{P}(\mathfrak{A}) \), there exists a ternary algebraic operation \( f \) depending on every variable. Further, taking into account the assumption \( 2 \in \mathcal{P}(\mathfrak{A}) \), we have either

\[
(13) \quad f(x, y, y) = y
\]
or

\[
(14) \quad f(x, y, y) = x.
\]

In the case (13) our statement is a consequence of Lemma 3. In the case (14) from the relations \( 3 \in \mathcal{P}(\mathfrak{A}) \) and \( 4 \in \mathcal{P}(\mathfrak{A}) \) we obtain, by Lemma 2, the assertion of the Lemma.

**Lemma 5.** If \( 2 \in \mathcal{P}(\mathfrak{A}) \), \( 3 \in \mathcal{P}(\mathfrak{A}) \) and \( 4 \in \mathcal{P}(\mathfrak{A}) \), then \( \mathfrak{A} = (G; \{g\} \circ F) \), where \( G \) is an at least two-element Boolean group, \( g(x, y, z) = x + y + z \) and all operations from \( F \) depend on at least five variables. Moreover, \( g \) is the only algebraic ternary operation depending on every variable.

**Proof.** Let \( f \) and \( g \) be ternary algebraic operations in the algebra \( \mathfrak{A} \) depending on every variable. Since \( 2 \in \mathcal{P}(\mathfrak{A}) \) and \( 4 \in \mathcal{P}(\mathfrak{A}) \), we have, by Lemma 3, the equations

\[
(15) \quad f(x, y, y) = f(y, x, y) = f(y, y, x) = x,
\]
and

\[
(16) \quad g(x, y, y) = g(y, x, y) = g(y, y, x) = x.
\]

Put

\[
(17) \quad h(x_1, x_2, x_3, x_4) = f(g(x_1, x_2, x_4), x_1, x_3).
\]

Since, by (15),

\[
(18) \quad h(x_1, x_2, x, x) = f(g(x_1, x_2, x), x, x) = g(x_1, x_2, x)
\]
and, by (16),

\[
(19) \quad h(x_1, y, x_3, y) = f(g(x_1, y, y), y, x_3) = f(x_1, y, x_3),
\]
we infer that the operation \( h \) depends on the variables \( x_1, x_2 \) and \( x_3 \). Therefore, by the assumption \( 4 \in \mathcal{P}(\mathfrak{A}) \), it does not depend on the variable \( x_4 \). Thus

\[
\begin{align*}
  h(x_1, x_2, x_3, x_4) &= h(x_1, x_2, x_3, x_3) = h(x_1, x_2, x_3, x_2).
\end{align*}
\]
Consequently, by (17), (18) and (19), we have the equations
\[ f(g(x_1, x_2, x_4, x_4, x_3) = g(x_1, x_2, x_3) = f(x_1, x_2, x_3), \]
which show that there is exactly one ternary algebraic operation \( g \) in the algebra \( \mathcal{A} \) depending on every variable. Moreover, this operation is symmetric and fulfills the equation
\[ g(g(x_1, x_2, x_4, x_4, x_3) = g(x_1, x_2, x_3). \]

Since \( \mathcal{T}(\mathcal{A}) \), the algebra \( \mathcal{A} \) can be written in the form \( \mathcal{A} = (G; \{ g \} \cup F) \), where all operations from \( F \) depends on at least five variables.

Let 0 be an element of \( G \). Put
\[ x + y = g(x, y, 0) \]
for all elements \( x \) and \( y \) from \( G \). From the symmetry of the operation \( g \) the equation \( x + y = y + x \) follows. Further, we have, according to (20),
\[ (x + y) + z = g(g(x, y, 0), z, 0) = g(g(x, y, 0), 0, z) = g(x, y, z) \]
and
\[ x + (y + z) = g(x, g(y, z, 0), 0) = g(x, g(y, z, 0), 0, x) = g(y, z, x) = g(x, y, z), \]
which proves the associativity law. Since, by (16), \( x + 0 = g(x, 0, 0) = x \), the element 0 is the zero-element. Further, by (16), \( x + x = g(x, x, 0) = 0 \), which shows that the set \( G \) is a Boolean group under the addition (21). Since \( 3 \in \mathcal{T}(\mathcal{A}) \), the set \( G \) contains at least two elements. Finally, from (22) we get the formula \( g(x, y, z) = x + y + z \), which completes the proof of the Lemma.

**Lemma 6.** Let \( G \) be a Boolean group and \( g(x, y, z) = x + y + z \) \((x, y, z \in G)\). Let \( f \) be an \( m \)-ary operation in \( G \), where \( m \geq 4 \). If for each system \( i_1, i_2, \ldots, i_m \) of indices satisfying the condition \( 1 < i_j < m - 1 \) \((j = 1, 2, \ldots, m)\) the operation \( f(x_{i_1}, x_{i_2}, \ldots, x_{i_m}) \) is algebraic in the algebra \((G; g)\), then there exists an \( m \)-ary algebraic operation \( f_0 \) in \((G; g)\) such that
\[ f(x_1, x_2, \ldots, x_m) = f_0(x_1, x_2, \ldots, x_m) \]
whenever at least two variables among \( x_1, x_2, \ldots, x_m \) are equal.

**Proof.** We note that each algebraic operation in the algebra \((G; g)\) is of the form
\[ h(x_1, x_2, \ldots, x_n) = x_{i_1} + x_{i_2} + \ldots + x_{i_k}, \]
where the indices \( i_1, i_2, \ldots, i_k \) are all different, \( 1 \leq i_j \leq n \) \((j = 1, 2, \ldots, k)\) and \( k \) is an odd integer.
Setting \( x_i = x_j \) \((i \neq j)\) into \( f(x_1, x_2, \ldots, x_m) \) we obtain an algebraic operation \( h_{ij} \) which, of course, does not depend on the variable \( x_i \). Consequently,

\[
h_{ij}(x_1, x_2, \ldots, x_m) = \sum_{s \in M(i, j)} x_s,
\]

where \( M(i, j) \) is a subset of the set \( \{1, 2, \ldots, m\} \) having an odd number of elements and satisfying the condition \( i \notin M(i, j) \).

Set \( u_s^{(e)} = x \) and \( u_k^{(e)} = y \) if \( k \neq s \) \((k = 1, 2, \ldots, m)\). For any index \( s, f(u_1^{(e)}, u_2^{(e)}, \ldots, u_m^{(e)}) \) is equal either to \( x \) or to \( y \). Let \( M \) be the set of all indices \( s \) for which the equation

\[
f(u_1^{(e)}, u_2^{(e)}, \ldots, u_m^{(e)}) = x
\]

holds. It is very easy to see that for \( s \neq i, j \) \((i \neq j)\) the relations \( s \in M(i, j) \) and \( s \notin M \) are equivalent. Consequently,

\[
M(i, j) \setminus \{i\} = M \setminus \{(i) \cup \{j\}\} \quad (i \neq j; \; i, j = 1, 2, \ldots, m).
\]

Suppose that \( i \neq j \). Since \( m \geq 4 \), we can find a pair of indices \( k, s \) \((k \neq s)\) different from \( i \) and \( j \). Put \( v_i = v_j = x \) and \( v_r = y \) if \( r \neq i, j \) \((r = 1, 2, \ldots, m)\). Then the equation

\[
h_{ij}(v_1, v_2, \ldots, v_m) = h_{ks}(v_1, v_2, \ldots, v_m)
\]

is true. Moreover, by (23),

\[
h_{ij}(v_1, v_2, \ldots, v_m) = \begin{cases} x & \text{if } j \in M(i, j), \\ y & \text{if } j \notin M(i, j) \end{cases}
\]

and

\[
h_{ks}(v_1, v_2, \ldots, v_m) = \begin{cases} y & \text{if both } i, j \in M(k, s) \text{ or both } i, j \notin M(k, s), \\ x & \text{in the opposite case}. \end{cases}
\]

Hence and from equation (25) it follows that \( j \in M(i, j) \) if and only if either \( i \in M(k, s) \) and \( j \notin M(k, s) \) or \( i \notin M(k, s) \) and \( j \in M(k, s) \). Now taking into account that all indices \( i, j, k, s \) are different we infer, in view of (24), that \( i \in M(k, s) \) if and only if \( i \in M \) and \( j \notin M(k, s) \) if and only if \( j \in M \). Consequently, \( j \in M(i, j) \) if and only if either \( i \in M \) and \( i \notin M \) or \( i \notin M \) and \( j \in M \). Hence and from (24) we obtain the equation

\[
M(i, j) = \begin{cases} M \setminus \{(i) \cup \{j\}\} & \text{if } i, j \in M \text{ or } i, j \notin M, \\ (M \setminus \{(i) \cup \{j\}\}) \cup \{j\} & \text{in the opposite case}. \end{cases}
\]
In particular, from this equation it follows that the set $M$ consists of an odd number of elements. Consequently, the operation

$$f_0(x_1, x_2, \ldots, x_m) = \sum_{s \in M} x_s$$

is algebraic in the algebra $(G; g)$. Further, by (23) and (26), we have the equation $f_0(x_1, x_2, \ldots, x_m) = h_{ij}(x_1, x_2, \ldots, x_m)$ whenever $x_i = x_j$. Thus the operation $f_0$ satisfies the assertion of the Lemma.

**Lemma 7.** Let $G$ be a Boolean group and $g(x, y, z) = x + y + z$ $(x, y, z \in G)$. Let $f$ be an $m$-ary operation on $G$ and $m \geq 4$. If for every system $i_1, i_2, \ldots, i_m$ of indices less than $m$ the operation $f(x_{i_1}, x_{i_2}, \ldots, x_{i_m})$ is algebraic in the algebra $(G; g)$, then there exists an $m$-ary operation $h$ such that $(G; g, f) = (G; g, h)$ and $h(x_1, x_2, \ldots, x_m) = x_1$ whenever at least two variables among $x_1, x_2, \ldots, x_m$ are equal.

**Proof.** By Lemma 6 there exists an $m$-ary algebraic operation $f_0$ in the algebra $(G; g)$ satisfying the equation

$$f(x_1, x_2, \ldots, x_m) = f_0(x_1, x_2, \ldots, x_m)$$

whenever at least two elements among $x_1, x_2, \ldots, x_m$ are identical. Without loss of generality we may assume that

$$f_0(x_1, x_2, \ldots, x_m) = x_1 + x_2 + \ldots + x_k,$$

where $k$ is an odd integer satisfying the inequality $1 \leq k \leq m$. Setting

$$h(x_1, x_2, \ldots, x_m) = f(x_1, x_2, \ldots, x_m) + x_2 + x_3 + \ldots + x_{k_2}$$

we have the equation

$$f(x_1, x_2, \ldots, x_m) = h(x_1, x_2, \ldots, x_m) + x_2 + x_3 + \ldots + x_k$$

which shows that $(G; g, f) = (G; g, h)$. Moreover,

$$h(x_1, x_2, \ldots, x_m) = f_0(x_1, x_2, \ldots, x_m) + x_2 + x_3 + \ldots + x_k$$

whenever at least two elements among $x_1, x_2, \ldots, x_m$ are equal. But, according to (27), the right-hand side of the last equation is equal to $x_1$, which completes the proof.

**Lemma 8.** Let $G$ be a Boolean group and $g(x, y, z) = x + y + z$ $(x, y, z \in G)$. Let $f$ be an $m$-ary operation in $G$ and $m \geq 4$. If for every system $h_1, h_2, \ldots, h_m$ of $(m-1)$-ary algebraic operations in the algebra $(G; g)$ the composition

$$f(h_1(x_1, x_2, \ldots, x_{m-1}), h_2(x_1, x_2, \ldots, x_{m-1}), \ldots, h_m(x_1, x_2, \ldots, x_{m-1}))$$
is an algebraic operation in the algebra \((G; g)\) and \(f(x_1, x_2, \ldots, x_m) = x_1\) whenever at least two variables among \(x_1, x_2, \ldots, x_m\) are identical, then \(f(x_1, x_2, \ldots, x_m) = x_1\) whenever \(x_1, x_2, \ldots, x_m\) belong to a subalgebra of the algebra \((G; g)\) generated by less than \(m\) elements.

**Proof.** Given a system \(h_1, h_2, \ldots, h_m\) of \((m-1)\)-ary algebraic operations in the algebra \((G; g)\) we put

\[
(28) \quad h_0(x_1, x_2, \ldots, x_{m-1}) = f(h_1(x_1, x_2, \ldots, x_{m-1}), h_2(x_1, x_2, \ldots, x_{m-1}), \ldots, h_m(x_1, x_2, \ldots, x_{m-1})).
\]

Of course, the operation \(h_0\) is algebraic in the algebra \((G; g)\). Since each algebraic operation in \((G; g)\) is a sum of an odd number of variables, we have the equations

\[
(29) \quad h_j(x_1, x_2, \ldots, x_{m-1}) = \sum_{s \in M_j} x_s \quad (j = 0, 1, \ldots, m),
\]

where \(M_0, M_1, \ldots, M_m\) are subsets of the set \(\{1, 2, \ldots, m-1\}\) consisting of an odd number of elements.

Put \(u_s^{(s)} = x\) and \(u_k^{(s)} = y\) if \(k \neq s\) \((k, s = 1, 2, \ldots, m-1)\). Since all binary algebraic operations in the algebra \((G; g)\) are trivial, we infer that, for every index \(s\) and every index \(j\), \(h_j(u_1^{(s)}, u_2^{(s)}, \ldots, u_{m-1}^{(s)})\) is equal either to \(x\) or to \(y\). Consequently, the system \(h_1(u_1^{(s)}, u_2^{(s)}, \ldots, u_{m-1}^{(s)}), h_2(u_1^{(s)}, u_2^{(s)}, \ldots, u_{m-1}^{(s)}), \ldots, h_m(u_1^{(s)}, u_2^{(s)}, \ldots, u_{m-1}^{(s)})\) contains at least two identical elements. Thus, by the assumption, we have the equation

\[
h_j(u_1^{(s)}, u_2^{(s)}, \ldots, u_{m-1}^{(s)}) = f(h_1(u_1^{(s)}, u_2^{(s)}, \ldots, u_{m-1}^{(s)}), h_2(u_1^{(s)}, u_2^{(s)}, \ldots, u_{m-1}^{(s)}), \ldots, h_m(u_1^{(s)}, u_2^{(s)}, \ldots, u_{m-1}^{(s)})).
\]

Hence and from (28) the equation

\[
(30) \quad h_0(u_1^{(s)}, u_2^{(s)}, \ldots, u_{m-1}^{(s)}) = h_1(u_1^{(s)}, u_2^{(s)}, \ldots, u_{m-1}^{(s)})
\]

\[(s = 1, 2, \ldots, m-1)\]

follows. Formula (29) implies the equivalence of the relation \(s \in M_j\) and the equation \(h_j(u_1^{(s)}, u_2^{(s)}, \ldots, u_{m-1}^{(s)}) = x\). Thus, by (30), we have the equation \(M_0 = M_1\), which, by (29), implies the equation \(h_0(x_1, x_2, \ldots, x_{m-1}) = h_1(x_1, x_2, \ldots, x_{m-1})\). Now the assertion of the Lemma is a simple consequence of formula (28).

**Lemma 9.** If \(2 \in \mathcal{F}(\mathcal{A})\), \(3 \in \mathcal{F}(\mathcal{A})\) and \(4 \in \mathcal{F}(\mathcal{A})\), then either \(\mathcal{F}(\mathcal{A})\) consists of all odd integers greater than 1 or \(\mathcal{F}(\mathcal{A})\) consists of all odd integers greater than 1 and all integers \(m \geq m\), where \(m \geq 5\). Moreover, the set \(\mathcal{F}(\mathcal{A})\) consists of all odd integers greater than 1 if and only if \(\mathcal{A} = (G; g)\), where \(G\) is an at least two-element Boolean group and \(g(x, y, z) = x+y+z\). The set \(\mathcal{F}(\mathcal{A})\) consists of all odd integers greater than 1 and all integers \(m \geq m\), where \(m \geq 5\) if and only if \(\mathcal{A} = (G; \{g\} \cup F)\), where \((G; g)\) is the algebra
defined in the preceding assertion, \( F \) contains an \( m \)-ary operation depending on every variable, all operations \( f \) from \( F \) depend on at least \( m \) variables and satisfy the equation \( f(x_1, x_2, \ldots, x_k) = x_1 \) whenever the elements \( x_1, x_2, \ldots, x_k \) belong to a subalgebra of the algebra \((G; g)\) generated by less than \( m \) elements.

Proof. Suppose that \( 2 \notin \mathcal{S}(\mathfrak{A}) \), \( 3 \notin \mathcal{S}(\mathfrak{A}) \) and \( 4 \notin \mathcal{S}(\mathfrak{A}) \). By Lemma 5 \( \mathfrak{A} = (G; \{g\} \cup F_0) \), where \( G \) is an at least two-element Boolean group, \( g(x, y, z) = x + y + z \) and all operations from \( F_0 \) depend on at least five variables. Moreover, \( g \) is the only ternary algebraic operation in \( \mathfrak{A} \). If all operations from \( F_0 \) are algebraic in the algebra \((G; g)\), then, of course, \( \mathfrak{A} = (G; g) \) and, consequently, \( \mathcal{S}(\mathfrak{A}) \) consists of all odd integers greater than 1.

Consider the remaining case and denote by \( m \) the smallest integer for which there exists an \( m \)-ary algebraic operation in \( \mathfrak{A} \) depending on every variable and which is not algebraic in the algebra \((G; g)\). Obviously, \( m \geq 5 \) and, by Lemmas 6, 7 and 8, \( \mathfrak{A} = (G; \{g\} \cup F) \), where \( F \) contains an \( m \)-ary operation depending on every variable, all operations \( f \) from \( F \) depend on at least \( m \) variables and satisfy the equation \( f(x_1, x_2, \ldots, x_k) = x_1 \) whenever the elements \( x_1, x_2, \ldots, x_k \) belong to a subalgebra of the algebra \((G; g)\) generated by less than \( m \) elements. To prove the Lemma it suffices to prove that for each such algebra \((G; \{g\} \cup F)\) the set \( \mathcal{S}(\mathfrak{A}) \) consists of all odd integers greater than 1 and all integers \( \geq m \).

The equation

\[(31) \quad \mathcal{S}(\mathfrak{A}) \cap \{2, 3, \ldots, m-1\} = \mathcal{S}((G; g)) \cap \{2, 3, \ldots, m-1\}\]

is obvious. Consequently, to prove our statement it suffices to prove that \( \mathcal{S}(\mathfrak{A}) \) contains all integers \( \geq m \). But this is a simple consequence of the inclusion \( \mathcal{S}(\mathfrak{A}) \supseteq \mathcal{S}(G; F) \) and Lemma 1. The Lemma is thus proved.

**Lemma 10.** Let \( r \geq 2 \), \( r \notin \mathcal{S}(\mathfrak{A}) \) and \( r+1 \notin \mathcal{S}(\mathfrak{A}) \). Let \( q \) be an \( r \)-ary algebraic operation in the algebra \( \mathfrak{A} \) depending on every variable. For each binary algebraic operation \( f \) there exists a subset \( Q_f \) of the set \( \{1, 2, \ldots, r\} \) such that

\[(32) \quad q(f(x_1, y_1), f(x_2, y_2), \ldots, f(x_r, y_r)) = q(u_1, u_2, \ldots, u_r), \]

where

\[(33) \quad u_j = \begin{cases} x_j & \text{if } j \in Q_f, \\ y_j & \text{if } j \notin Q_f. \end{cases}\]

The correspondence between binary operations \( f \) and subsets \( Q_f \) is one-to-one. Moreover, for each pair \( f, g \) of binary algebraic operations we have the equation

\[(34) \quad q(g(x_1, y_1), z_1), f(g(x_2, y_2), z_2), \ldots, q(g(x_r, y_r), z_r) = q(v_1, v_2, \ldots, v_r)\]
where

\[ v_j = \begin{cases} 
    x_j & \text{if } j \in Q_f \cap Q_g, \\
    y_j & \text{if } j \in Q_f \setminus Q_g, \\
    z_j & \text{if } j \notin Q_f.
\end{cases} \]

(35)

\[ h_f(x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_r) = q(f(x_1, y_1), f(x_2, y_2), \ldots, f(x_r, y_r)). \]

(36)

Proof. Given a binary operation \( f \) we put

\[ h_f(x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_r) = q(f(x_1, y_1), f(x_2, y_2), \ldots, f(x_r, y_r)). \]

(36)

From the equation

\[ h_f(x_1, x_2, \ldots, x_r, x_1, x_2, \ldots, x_r) = q(x_1, x_2, \ldots, x_r) \]

it follows that for each index \( j \) (\( j = 1, 2, \ldots, r \)) the operation \( h_f \) depends on at least one variable \( x_j \) or \( y_j \). If the operation \( h_f \) would depend on a pair \( x_j, y_j \) simultaneously, then the \( (r+1) \)-ary operation

\[ h(x_1, x_2, \ldots, x_r, y_j) = q(x_1, x_2, \ldots, x_{j-1}, f(x_j, y_j), x_{j+1}, \ldots, x_r) \]

would depend on every variable. But this contradicts the assumption \( r+1 \notin \mathcal{S}(2) \). Thus for each index \( j \) (\( j = 1, 2, \ldots, r \)) the operation \( h_f \) depends on exactly one variable \( x_j \) or \( y_j \). Let \( Q_f \) be the set of all indices \( j \) for which \( h_f(x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_r) \) depends on the variable \( x_j \). If the elements \( u_1, u_2, \ldots, u_r \) are defined by formula (33), then we have the equation

\[ h_f(x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_r) = h_f(u_1, u_2, \ldots, u_r, u_1, u_2, \ldots, u_r), \]

which, in view of (36), implies equation (32).

It is obvious that the set \( Q_f \) uniquely determines the operation \( h_f \).

Since, by (36),

\[ h_f(x, x, \ldots, x, y, y, \ldots, y) = q(f(x, y), f(x, y), \ldots, f(x, y)) = f(x, y), \]

we infer that the correspondence between binary operations \( f \) and the sets \( Q_f \) is one-to-one.

Further, by (32) and (33), we have the equation

(37) \[ q\left( f\left( g(x_1, y_1), z_1 \right), f\left( g(x_2, y_2), z_2 \right), \ldots, f\left( g(x_r, y_r), z_r \right) \right) = q(w_1, w_2, \ldots, w_r), \]

where

\[ w_j = \begin{cases} 
    g(x_j, y_j) & \text{if } j \in Q_f, \\
    z_j & \text{if } j \notin Q_f.
\end{cases} \]

On the other hand, according to (36), we have the equation

(38) \[ q(w_1, w_2, \ldots, w_r) = h_g(a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_r), \]
where

\[
(a_j = \begin{cases} x_j & \text{if } j \in Q_f, \\ z_j & \text{if } j \notin Q_f, \end{cases} \quad b_j = \begin{cases} y_j & \text{if } j \in Q_f, \\ z_j & \text{if } j \notin Q_f. \end{cases}
\]

Applying once more formula (32) we get the equation

\[
h_g(a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_r) = q(v_1, v_2, \ldots, v_r),
\]

where

\[
v_j = \begin{cases} a_j & \text{if } j \in Q_g, \\ b_j & \text{if } j \notin Q_g. \end{cases}
\]

Hence and from (39) it follows that the elements \(v_1, v_2, \ldots, v_r\) satisfy condition (35). Equation (34) is now a simple consequence of equations (37), (38) and (40). The Lemma is thus proved.

**Lemma 11.** Let \(2 \in \mathcal{S}(\mathcal{A})\) and \(\mathcal{S}(\mathcal{A}) \neq \{2, 3, \ldots\}\). The set of all binary algebraic operations in \(\mathcal{A}\) is a finite Boolean algebra under the operations

\[
(f')(x, y) = f(y, x),
\]

\[
(f \triangleleft g)(x, y) = f(g(x, y), y),
\]

\[
(f \triangleright g)(x, y) = f(x, g(x, y)).
\]

The unit element \(1\) and the neutral element \(0\) are trivial operations

\[
1(x, y) = x, \quad 0(x, y) = y.
\]

Further, for all binary operations \(f\) in the algebra \(\mathcal{A}\) the equation

\[
f\{f(x, y), f(z, u)\} = f(x, u)
\]

holds. Moreover, if \(f \triangleleft g = 0\), then

\[
f(g(x, y), z) = f(y, z)
\]

and

\[
f(x, g(y, z)) = g(y, f(x, z)).
\]

**Proof.** By the definitions (41), (42) and (43) we have the equation

\[
(f \triangleleft g)'(x, y) = \left[f\left(g(x, y), y\right)\right]' = f\left(g(y, x), x\right) = f'(x, g'(x, y)) = (f' \triangleright g')(x, y).
\]

Since \(\mathcal{S}(\mathcal{A}) \neq \{2, 3, \ldots\}\) and \(2 \in \mathcal{S}(\mathcal{A})\), there exists an integer \(r \geq 2\) such that \(r \in \mathcal{S}(\mathcal{A})\) and \(r + 1 \notin \mathcal{S}(\mathcal{A})\). Let \(q\) be an \(r\)-ary algebraic operation depending on every variable. By Lemma 10 the operation \(q\) induces a one-to-one correspondence between binary algebraic operations \(f\) and
subsets \( Q_f \) of the set \( \{1, 2, \ldots, r\} \) such that equations (32) and (34) hold. To prove that the set of all binary algebraic operations in the algebra \( \mathcal{A} \) is a Boolean algebra it suffices, by (47), to prove the formulas

\[
\begin{align*}
Q_f' &= Q_f', \\
Q_{f \cap g} &= Q_f \cap Q_g,
\end{align*}
\]

where \( Q_f' = \{1, 2, \ldots, r\} \setminus Q_f \). Indeed, these formulas show that the correspondence between binary operations \( f \) and sets \( Q_f \) preserves Boolean operations and, consequently, is a Boolean isomorphism.

Taking into account (41) we have the equation

\[
q(f(x_1, y_1), f(x_2, y_2), \ldots, f(x_r, y_r)) = q(f'(y_1, x_1), f'(y_2, x_2), \ldots, f'(y_r, x_r)).
\]

By Lemma 10 the left-hand side of the last equation is equal to 
\( q(u_1, u_2, \ldots, u_r) \), where the quantities \( u_1, u_2, \ldots, u_r \) are defined by formula (33) and the right-hand side is equal to 
\( q(t_1, t_2, \ldots, t_r) \), where \( t_j = y_j \) if \( j \in Q_f \) and \( t_j = x_j \) in the opposite case. Hence we get formula (48).

Further, from (32), (33) and (42) we get the equation

\[
q(f(g(x_1, y_1), y_1), f(g(x_2, y_2), y_2), \ldots, f(g(x_r, y_r), y_r)) = q(a_1, a_2, \ldots, a_r),
\]

where \( a_j = x_j \) if \( j \in Q_{f \cap g} \) and \( a_j = y_j \) in the opposite case. On the other hand, by formulas (34) and (35), we have the equation

\[
q(f(g(x_1, y_1), y_1), f(g(x_2, y_2), y_2), \ldots, f(g(x_r, y_r), y_r)) = q(b_1, b_2, \ldots, b_r),
\]

where \( b_j = x_j \) if \( j \in Q_f \cap Q_g \) and \( b_j = y_j \) in the opposite case. Hence we get formula (49), which completes the proof that the set of all binary algebraic operations is a Boolean algebra under the operations (41), (42) and (43). The proof that the operations \( 1(x, y) \) and \( 0(x, y) \) are a unit element and a neutral element respectively is obvious.

Now we proceed to the proof of formula (44). From (34) and (35) it follows that

\[
q(f(f(x_1, y_1), z_1), f(f(x_2, y_2), z_2), \ldots, f(f(x_r, y_r), z_r)) = q(v_1, v_2, \ldots, v_r),
\]

where \( v_j = x_j \) if \( j \in Q_f \) and \( v_j = z_j \) in the opposite case. Hence and from (32) and (33) we get the equation

\[
q(f(f(x_1, y_1), z_1), f(f(x_2, y_2), z_2), \ldots, f(f(x_r, y_r), z_r)) = q(f(x_1, z_1), f(x_2, z_2), \ldots, f(x_r, z_r)).
\]
Setting \( x_j = x, \ y_j = y, \ z_j = z \ (j = 1, 2, \ldots, r) \) into the last equation and taking into account the formula \( q(x, x, \ldots, x) = x \), we get the equation

\[
f(f(x, y), z) = f(x, z)
\]

for all binary algebraic operations \( f \). Thus, by (41),

\[
f(f(x, y), f(z, u)) = f(x, f(z, u)) = f'(f'(u, z), x) = f'(u, x) = f(x, u).
\]

Formula (44) is thus proved.

Suppose that \( f \circ g = 0 \). Consequently, the sets \( Q_f \) and \( Q_g \) are disjoint. Thus, by (34) and (35),

\[
q\{f(g(x_1, y_1), z_1), f(g(x_2, y_2), z_2), \ldots, f(g(x_r, y_r), z_r)\} = q(v_1, v_2, \ldots, v_r),
\]

where \( v_j = y_j \) if \( j \in Q_f \) and \( v_j = z_j \) in the opposite case. Hence and from (32) and (33) we obtain the equation

\[
q\{f(g(x_1, y_1), z_1), f(g(x_2, y_2), z_2), \ldots, f(g(x_r, y_r), z_r)\}
= q\{f(y_1, z_1), f(y_2, z_2), \ldots, f(y_r, z_r)\}.
\]

Setting \( x_j = x, \ y_j = y \) and \( z_j = z \ (j = 1, 2, \ldots, r) \) into the last equation we obtain formula (45).

Further, we have, according to (34) and (35), the equations

\[
q\{f'(g(y_1, z_1), x_1), f'(g(y_2, z_2), x_2), \ldots, f'(g(y_r, z_r), x_r)\}
= q(w_1, w_2, \ldots, w_r),
\]

where

\[
w_j = \begin{cases} x_j & \text{if } j \notin Q_f' \\ y_j & \text{if } j \in Q_f' \cap Q_g \\ z_j & \text{if } j \in Q_f' \setminus Q_g \end{cases}
\]

and

\[
q\{g'(f(x_1, z_1), y_1), g'(f(x_2, z_2), y_2), \ldots, g'(f(x_r, z_r), y_r)\}
= q(t_1, t_2, \ldots, t_r),
\]

where

\[
t_j = \begin{cases} x_j & \text{if } j \in Q_g' \cap Q_f \\ y_j & \text{if } j \notin Q_g' \\ z_j & \text{if } j \in Q_g' \setminus Q_f \end{cases}
\]

We assumed that the sets \( Q_f \) and \( Q_g \) are disjoint. Consequently,

\[
Q_f = Q_g' \cap Q_f, \quad Q_f' \cap Q_g = Q_g \quad \text{and} \quad Q_f' \setminus Q_g = Q_g' \setminus Q_f.
\]
Hence and from (51) and (53), in view of formula (48), the equation \( w_j = t_j \) \((j = 1, 2, \ldots, r)\) follows. Consequently, from (41), (50) and (52) we get the equation
\[
g\left(f(x_1, g(y_1, z_1)), f(x_2, g(y_2, z_2)), \ldots, f(x_r, g(y_r, z_r))\right)
\]
\[
= g\left(g(y_1, f(x_1, z_1)), g(y_2, f(x_2, z_2)), \ldots, g(y_r, f(x_r, z_r))\right).
\]

Now, setting \( x_j = x, \ y_j = y \) and \( z_j = z \) \((j = 1, 2, \ldots, r)\) into the last equation we obtain formula (46). The Lemma is thus proved.

**Corollary.** Let \( 2 \in \mathcal{F}(\mathcal{A}) \) and \( \mathcal{F}(\mathcal{A}) \neq \{2, 3, \ldots\} \). If \( f \) is a binary algebraic operation in \( \mathcal{A} \) and \( g \) an \( n \)-ary not necessarily algebraic operation in \( \mathcal{A} \) and
\[
h(x_1, x_2, \ldots, x_n) = f(g(x_2, x_1, \ldots, x_n), g(x_1, x_2, \ldots, x_n)),
\]
them
\[
g(x_1, x_2, \ldots, x_n) = f(h(x_2, x_1, \ldots, x_n), h(x_1, x_2, \ldots, x_n)).
\]

In particular, the operations \( g \) and \( h \) are either both algebraic or both non-algebraic.

Indeed, from formula (44) we get the equation
\[
f(h(x_2, x_1, \ldots, x_n), h(x_1, x_2, \ldots, x_n))
\]
\[
= f\left(f\left(g(x_1, x_2, \ldots, x_n), g(x_2, x_1, \ldots, x_n)\right), f\left(g(x_2, x_1, \ldots, x_n), g(x_1, x_2, \ldots, x_n)\right)\right)
\]
\[
= f\left(g(x_1, x_2, \ldots, x_n), g(x_1, x_2, \ldots, x_n)\right) = g(x_1, x_2, \ldots, x_n).
\]

**Lemma 12.** Let \( 2 \in \mathcal{F}(\mathcal{A}) \) and \( \mathcal{F}(\mathcal{A}) \neq \{2, 3, \ldots\} \). Then each ternary algebraic operation \( f \) satisfies the equation
\[
f(f(x, y, z), x, x) = x.
\]

**Proof.** Put
\[
h(x, y, z) = f(f(x, y, z), x, x) \quad \text{and} \quad g(x, y) = f(x, y, y).
\]

From Lemma 11 (formula (44)) we get the equation
\[
(54) \quad h(x, y, y) = g(g(x, y), x) = g(x, x) = x.
\]

Suppose that the operation \( h \) does not depend on every variable. Then formula (54) implies the equation \( h(x, y, z) = x \), which gives the assertion of the Lemma.

If the operation \( h \) depends on every variable, then, of course, \( 3 \in \mathcal{F}(\mathcal{A}) \). Since, in addition, \( 2 \in \mathcal{F}(\mathcal{A}) \), we infer, in view of (54) and Lemma 2, that \( \mathcal{F}(\mathcal{A}) = \{2, 3, \ldots\} \), which contradicts the assumption \( \mathcal{F}(\mathcal{A}) \neq \{2, 3, \ldots\} \). The Lemma is thus proved.
Lemma 13. Let \( 2 \in \mathcal{P}(\mathfrak{A}) \) and \( \mathcal{P}(\mathfrak{A}) \neq \{2, 3, \ldots\} \). Then each ternary algebraic operation in \( \mathfrak{A} \) is generated by binary algebraic operation in \( \mathfrak{A} \), i.e. is a composition of binary algebraic operations.

Proof. Contrary to this let us suppose that there exists a ternary algebraic operation \( g \) which is not generated by binary algebraic operations. Put
\[
(55) \quad f_1(x, y, z) = g(x, y, z)
\]
and
\[
(56) \quad h_1(x, y, z) = f_1(g(y, x, z), g(x, y, z)).
\]

By virtue of the Corollary to Lemma 11, the operation \( h_1 \) is not generated by binary algebraic operations. Moreover, according to (44), (55) and Lemma 12, we have the equation
\[
(57) \quad h_1(x, y, z) = f_1(g(y, x, z), f_1(x, y)) = f_1(g(y, x, z), g(x, y, z)) = g(g(y, x, z), y, y) = y.
\]

Set
\[
(58) \quad f_2(x, y) = g(y, y, x)
\]
and
\[
(59) \quad h_2(x, y, z) = f_2(h_1(z, x, y), h_1(z, y, x)).
\]

By Corollary to Lemma 11, the operation \( h_2 \) is not generated by binary algebraic operations. Further, from (57) and (59) it follows that
\[
(60) \quad h_2(z, y, x) = f_2(h_1(z, y, y), h_1(z, y, y)) = h_1(z, y, y) = y.
\]

Let us introduce an auxiliary operation \( h_0(x, y, z) = h_1(z, y, x) \).

Since, by (56) and (58),
\[
(61) \quad h_0(x, y, y) = h_1(y, y, x) = f_1(g(y, y, y), g(y, y, x)) = g(y, y, x) = f_2(x, y),
\]
we have, by Lemma 12, the equation
\[
f_2(h_1(y, x, y), y) = f_2(h_0(y, x, y), y) = h_0(h_0(y, x, y), y, y) = y.
\]

Hence and from (44), (59) and (61) we get the equation
\[
(62) \quad h_2(x, y, y) = f_2(h_1(y, x, y), h_1(y, y, x)) = f_2(h_1(y, x, y), f_2(x, y)) = f_2(h_1(y, x, y), y) = y.
\]
If the operation \( h_2(y, x, y) \) does not depend on both variables \( x \) and \( y \) and, consequently, is a trivial operation, then, by (60) and (62), the algebra \( (A; h_2) \) contains no binary operation depending on every variable. Thus, by Lemma 3, \( \mathcal{P}(\langle A; h_2 \rangle) = \{ s : s \geq 3 \} \) and, consequently, \( \mathcal{P}(\mathfrak{U}) = \{2, 3, \ldots\} \), which contradicts the assumption \( \mathcal{P}(\mathfrak{U}) \neq \{2, 3, \ldots\} \). Therefore the operation \( h_2(y, x, y) \) depends on both variables \( x \) and \( y \). Put

\[
f(x, y) = h_2(y, x, y).
\]

(63)

Since \( \mathcal{P}(\mathfrak{U}) \neq \{2, 3, \ldots\} \), there exists an integer \( r \geq 2 \) such that \( r \in \mathcal{P}(\mathfrak{U}) \) and \( r + 1 \notin \mathcal{P}(\mathfrak{U}) \). Let \( q \) be an \( r \)-ary algebraic operation depending on every variable. By Lemma 10 there exists a subset \( Q_f \) of the set \( \{1, 2, \ldots, r\} \) such that formula (32) holds. Since the operation \( f \) is non-trivial, we have the inequality \( Q_f \neq \{1, 2, \ldots, r\} \). Without loss of generality we may assume that \( 1 \notin Q_f \). From formula (32) we get the equation

\[
q(f(x_1, y), x_2, \ldots, x_r) = q(y, x_2, \ldots, x_r).
\]

(64)

We define an \((r+1)\)-ary algebraic operation by means of the formula

\[
h(x_1, x_2, \ldots, x_{r+1}) = q(h_2(x_1, x_{r+1}, x_2), x_2, \ldots, x_r).
\]

(65)

Setting \( x_1 = x_{r+1} = y \) into (65) we obtain, in view of (60), the equation

\[
h(y, x_2, \ldots, x_r, y) = q(h_2(y, x_2, x_2, \ldots, x_r) = q(y, x_2, \ldots, x_r),
\]

which shows that the operation \( h \) depends on the variables \( x_2, x_3, \ldots, x_r \) and on at least one of the variables \( x_1, x_{r+1} \).

Suppose that the operation \( h \) depends on the variable \( x_{r+1} \) and does not depend on the variable \( x_1 \). Then, by (63) and (64), we have the equation

\[
h(x_1, x_2, \ldots, x_{r+1}) = h(x_2, x_2, \ldots, x_{r-1}) = q(h_2(x_2, x_{r+1}, x_2), x_2, \ldots, x_r
\]

\[
= q(f_2(x_{r+1}, x_2), x_2, \ldots, x_r) = q(x_2, x_2, \ldots, x_r),
\]

which shows that the operation \( h \) does not depend on the variable \( x_{r+1} \). But this contradicts the assumption.

Further, suppose that the operation \( h \) depends on the variable \( x_1 \) and does not depend on the variable \( x_{r+1} \). In this case, according to (62), we have the equation

\[
h(x_1, x_2, \ldots, x_{r+1}) = h(x_1, x_2, \ldots, x_r, x_2)
\]

\[
= q(h_2(x_1, x_2, x_2), x_2, \ldots, x_r) = q(x_2, x_2, \ldots, x_r),
\]

which contradicts the assumption.
which shows that the operation \( h \) does not depend on the variable \( x_1 \). But this gives a contradiction. Consequently, the \((r+1)\)-ary operation \( h \) depends on every variable, which contradicts the assumption \( r+1 \notin \mathcal{A}(\mathfrak{A}) \). The Lemma is thus proved.

**Lemma 14.** If \( 2 \in \mathcal{A}(\mathfrak{A}) \) and \( \mathcal{A}(\mathfrak{A}) \neq \{2, 3, \ldots\} \), then \( (A; A^{(2)}) \) is a diagonal algebra.

**Proof.** By Lemma 11 the class \( A^{(2)} \) of binary algebraic operations is a finite Boolean algebra under the operations (41), (42) and (43). Denoting by \( d_1, d_2, \ldots, d_n \) the atoms of this Boolean algebra we have the obvious equation \( (A; A^{(2)}) = (A; d_1, d_2, \ldots, d_n) \).

Each operation \( d_j \) \( (j = 1, 2, \ldots, n) \) induces a congruence relation in the set \( A \) as follows: \( a \sim_j b \) if and only if \( d_j(a, x) = d_j(b, x) \) for all \( x \in A \). If \( a \sim_j b \) for all \( j = 1, 2, \ldots, n \), then \( a = b \). Indeed, by the definition of the unit element,

\[
a = 1(a, x) = (\bigcup_{j=1}^{n} d_j)(a, x) = d_1 \left( a, d_2(a, \ldots, d_n(a, x)) \ldots \right)
\]

\[
= d_1 \left( b, d_2(b, \ldots, d_n(b, x)) \ldots \right) = (\bigcup_{j=1}^{n} d_j)(b, x) = 1(b, x) = b.
\]

Now we shall prove that for every system \( a_1, a_2, \ldots, a_n \) of elements of \( A \) there exists an element \( c \) of \( A \) such that \( c \sim_j a_j \) \( (j = 1, 2, \ldots, n) \).

Put

\[
c = d_1 \left( a_1, d_2 \left( a_2, \ldots, d_{n-1} \left( a_{n-1}, d_n(a_n, a_n) \right) \ldots \right) \right).
\]

Since the atoms \( d_1, d_2, \ldots, d_n \) are disjoint, we infer, by formula (46) of Lemma 11, that for each index \( j \) \( (j = 1, 2, \ldots, n) \) there exists an element \( b_j \in A \) such that \( c = d_j(a_j, b_j) \). Hence and from (44) for all \( x \in A \) the equation

\[
d_j(c, x) = d_j \left( d_j(a_j, b_j), x \right) = d_j(a_j, x)
\]

follows. Consequently, \( c \sim_j a_j \) \( (j = 1, 2, \ldots, n) \).

Taking into account these properties of the congruence relations \( \sim_j \), we infer, in view of the factorization theorem (see [1], Theorem 4, Chapter VI, § 2), that the set \( A \) is a Cartesian product \( A = A_1 \times A_2 \times \ldots \times A_n \). Moreover, denoting by \( a_k \) and \( b_k \) the elements of \( A_k \) \( (k = 1, 2, \ldots, n) \) and setting

\[
a = \langle a_1, a_2, \ldots, a_n \rangle, \quad b = \langle b_1, b_2, \ldots, b_n \rangle,
\]

we have the relation \( a \sim_j b \) if and only if \( a_j = b_j \). Since, by (44) and (45),

\[
d_k \left( d_j(a, b), x \right) = \begin{cases} d_j(a, x) & \text{if } k = j, \\ d_k(b, x) & \text{if } k \neq j, \end{cases}
\]
we have the relations \( d_j(a, b) \sim a \) and \( d_j(a, b) \sim b \) if \( k \neq j \). Hence it follows that

\[
d_j(\langle a_1, a_2, \ldots, a_n \rangle, \langle b_1, b_2, \ldots, b_n \rangle) = \langle b_1, b_2, \ldots, b_{j-1}, a_j, b_{j+1}, \ldots, b_n \rangle.
\]

In other words, \((A; d_1, d_2, \ldots, d_n)\) is a diagonal algebra. The Lemma is thus proved.

**Lemma 15.** Let \( \mathbb{A} \) be a diagonal algebra, \( m \geq 4 \) and \( f \) an \( m \)-ary operation in \( \mathbb{A} \) not necessarily algebraic. If for each system \( i_1, i_2, \ldots, i_m \) of indices less than \( m \) the operation \( f(x_{i_1}, x_{i_2}, \ldots, x_{i_m}) \) is algebraic in the algebra \( \mathbb{A} \), then there exists an \( m \)-ary algebraic operation \( g \) in \( \mathbb{A} \) such that

\[
f(x_1, x_2, \ldots, x_m) = g(x_1, x_2, \ldots, x_m)
\]

whenever at least two variables among \( x_1, x_2, \ldots, x_m \) are equal.

**Proof.** Let \( \mathbb{A} = (A_1 \times A_2 \times \ldots \times A_n; d_1, d_2, \ldots, d_n) \), where the fundamental binary operations \( d_1, d_2, \ldots, d_n \) are defined, by formula (1). It is very easy to see that each algebraic operation is of the form

\[
d_1(\langle x_{k_1}, d_2(\langle x_{k_2}, \ldots, d_n(\langle x_{k_n}, x_{k_1} \rangle) \ldots \rangle),
\]

where the system \( k_1, k_2, \ldots, k_n \) of indices is uniquely determined.

Suppose that the \( m \)-ary operation \( f \) satisfies the assumptions of the Lemma. Substituting \( x_i \) by \( x_j \) \((i \neq j; i, j = 1, 2, \ldots, m)\) in \( f(x_1, x_2, \ldots, x_m) \) we obtain an algebraic operation \( f_{ij}(x_1, x_2, \ldots, x_m) \) which, of course, does not depend on the variable \( x_i \). The operation \( f_{ij} \) can be written in the form

\[
f_{ij}(x_1, x_2, \ldots, x_m) = d_1(\langle x_{k_1(i,j)}, d_2(\langle x_{k_2(i,j)}, \ldots, d_n(\langle x_{k_n(i,j)}, x_{k_1(i,j)} \rangle) \ldots)\rangle)
\]

where \( 1 \leq k_s(i, j) \leq m, k_s(i, j) \neq i \) \((s = 1, 2, \ldots, n)\).

First we assume that for all pairs \( i, j \) \((i \neq j)\) the equation \( k_1(i, j) = j \) holds. Taking into account the inequality \( m \geq 4 \), we get, in view of (66) the equations

\[
f(x_2, x_2, x_3, x_4, \ldots, x_m) = d_1(x_2, d_2(x_{k_2(1,2)}, \ldots, d_n(x_{k_n(1,2)}, x_{k_n(1,2)}))\ldots),
\]

\[
f(x_1, x_2, x_4, x_4, \ldots, x_m) = d_1(x_4, d_2(x_{k_2(3,4)}, \ldots, d_n(x_{k_n(3,4)}, x_{k_n(3,4)}))\ldots).
\]

Setting \( x_3 = x_1 \) into (67) and \( x_1 = x_2 \) into (68) we get identical expressions. But the right-hand side of (67) is then equal to \( d_1(x_2, h) \), where \( h \) is an algebraic operation. Similarly, the right-hand side of (68) is equal to \( d_1(x_4, g) \), where \( g \) is an algebraic operation. But, by virtue of (1), the equation \( d_1(x_2, h) = d_1(x_4, g) \) never holds for \( x_2 \neq x_4 \). Consequently, there exists a pair \( i, j \) \((i \neq j)\) for which the inequality \( k_1(i, j) \neq j \) holds.
Without loss of generality we may assume that $k_1(1, 2) = 3$. Setting $x_i = x_j (i \neq j)$ into $f_{12}$ and $x_1 = x_2$ into $f_{ij}$ we obtain the same expressions. But, by (66), the operation $f_{12}$ is then of the form $d_i(x_3, h)$ if $i \neq 3$ and of the form $d_i(x_j, h)$ if $i = 3$, where $h$ is an algebraic operation. Similarly, the operation $f_{ij}$ is of the form $d_i(x_{k_1(i,j)}, g)$ if $k_1(i, j) \neq 1$ and of the form $d_i(x_3, g)$ if $k_1(i, j) = 1$. Hence we get the equalities $k_1(3, j) = j$ and $k_1(i, j) = 3$ if $i \neq 3$. Since the system $d_1, d_2, \ldots, d_n$ of fundamental operations can be arbitrarily indexed, the iteration of the previous reasoning leads to the existence of a system of indices $r_1, r_2, \ldots, r_m$ such that $1 \leq r_s \leq m$ ($s = 1, 2, \ldots, n$), $k_s(r_s, j) = j$ ($s = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m; j \neq r_s$) and $k_s(i, j) = r_s$ ($s = 1, 2, \ldots, n; i \neq j, i \neq r_s; j = 1, 2, \ldots, m$). Put

$$g(x_1, x_2, \ldots, x_m) = d_1(x_{r_1}, d_2(x_{r_2}, \ldots, d_n(x_{r_n}, x_{r_n})) \ldots).$$

The operation $g$ is algebraic and, moreover, by (66),

$$g(x_1, x_2, \ldots, x_m) = f_{ij}(x_1, x_2, \ldots, x_m)$$

whenever $x_i = x_j (i \neq j)$. The Lemma is thus proved.

**Lemma 16.** Let $(A; d)$ be a diagonal algebra, $m \geq 4$ and $f$ an $m$-ary operation in $A$ not necessarily algebraic. If for each system $i_1, i_2, \ldots, i_m$ of indices less than $m$ the operation $f(x_{i_1}, x_{i_2}, \ldots, x_{i_m})$ is algebraic in the algebra $(A; d)$, then there exists an $m$-ary operation $h$ for which $h(x_1, x_2, \ldots, x_m) = x_1$ whenever at least two variables among $x_1, x_2, \ldots, x_m$ are equal and $(A; d, f) = (A; d, h)$.

**Proof.** We use the notation introduced in the proof of the previous Lemma. By Lemma 15 there exists an $m$-ary algebraic operation $g$ such that $f(x_1, x_2, \ldots, x_m) = g(x_1, x_2, \ldots, x_m)$ whenever at least two variables among $x_1, x_2, \ldots, x_m$ are identical. Without loss of generality we may assume that

$$g(x_1, x_2, \ldots, x_m) = g_0(x_1, x_2, \ldots, x_k),$$

where $1 \leq k \leq m$ and the operation $g_0$ depends on every variable.

If $k = 1$, then setting $h = f$ we get the assertion of the Lemma. Now we shall prove that the case $k > 1$ can be reduced to the case $k = 1$.

Suppose that $k > 1$. The fundamental operations (1) can be indexed in such a way that

$$g_0(x_1, x_2, \ldots, x_k) = d_1(x_{r_1}, d_2(x_{r_2}, \ldots, d_n(x_{r_n}, x_{r_n})) \ldots),$$

where

$$r_1 = r_2 = \ldots = r_p = k, \quad r_{p+1} = r_{p+2} = \ldots = r_q = k-1,$$

$$r_j < k-1 \quad (j = q+1, q+2, \ldots, n)$$

and $1 \leq p < q \leq n$. 


Put
\[ f_1(x, y) = d_1\left(x, d_2(x, \ldots, d_p(x, y))\right), \]
\[ g_1(x_1, x_2, \ldots, x_k) = f_1\left(g_0(x_1, x_2, \ldots, x_{k-2}, x_k, x_{k-1}), g_0(x_1, x_2, \ldots, x_k)\right), \]
\[ h_1(x_1, x_2, \ldots, x_m) = f_1\left(f(x_1, x_2, \ldots, x_{k-2}, x_k, x_{k-1}, x_{k+1}, \ldots, x_m), f(x_1, x_2, \ldots, x_m)\right). \]

By Corollary to Lemma 11 we have the equation \((A; d, f) = (A; d, h_1)\). Moreover, \(h_1(x_1, x_2, \ldots, x_m) = g_1(x_1, x_2, \ldots, x_k)\) whenever at least two variables among \(x_1, x_2, \ldots, x_m\) are equal. By (69), (70), (71) and (72) we have the equation
\[ g_1(x_1, x_2, \ldots, x_k) = f_1\left(d_1\left(x_{k-1}, d_2\left(x_{k-1}, \ldots, d_p\left(x_{k-1}, d_{k+1}(x_k, \ldots, d_n\left(x_{r_n}, x_{r_n})\right))\right)\right)\right), \]
\[ d_1\left(x_k, d_2(x_k, \ldots, d_p\left(x_k, d_{p+1}(x_{k-1}, \ldots, d_n\left(x_{r_n}, x_{r_n})\right))\right)\right) \]
\[ f_1\left(f_1\left(x_{k-1}, d_{p+1}(x_k, d_{p+2}(x_k, \ldots, d_n\left(x_{r_n}, x_{r_n})\right))\right)\right), \]
\[ f_1\left(x_k, d_{p+1}\left(x_{k-1}, d_{p+2}(x_{k-1}, \ldots, d_n\left(x_{r_n}, x_{r_n})\right)\right)\right). \]

Hence and from (44) we get the formula
\[ g_1(x_1, x_2, \ldots, x_k) = f_1\left(x_{k-1}, d_{p+1}\left(x_{k-1}, d_{p+2}(x_{k-1}, \ldots, d_n\left(x_{r_n}, x_{r_n})\right)\right)\right), \]
which, according to (70), proves that the operation \(g_1\) does not depend on the variable \(x_k\). Setting \(g_2(x_1, x_2, \ldots, x_{k-1}) = g_1(x_1, x_2, \ldots, x_k)\), we infer that \(h_1(x_1, x_2, \ldots, x_m) = g_2(x_1, x_2, \ldots, x_{k-1})\) whenever at least two variables among \(x_1, x_2, \ldots, x_m\) are equal. By a consecutive iteration of this procedure we can reduce our problem to the case \(k = 1\), which completes the proof of the Lemma.

**Lemma 17.** Let \(f\) and \(g\) be a pair of \(m\)-ary algebraic operations in a diagonal algebra, where \(m \geq 3\). If the equation \(f(x_1, x_2, \ldots, x_m) = g(x_1, x_2, \ldots, x_m)\) holds whenever \(x_1 = x_2\) or \(x_1 = x_3\), then \(f = g\).

**Proof.** The operations \(f\) and \(g\) are compositions of fundamental operations \(d_1, d_2, \ldots, d_n\) defined by formula (1). Moreover, the representation
\[ f(x_1, x_2, \ldots, x_m) = d_1\left(x_{r_1}, d_2(x_{r_2}, \ldots, d_n(x_{r_n}, x_{r_n})\ldots)\right), \]
\[ g(x_1, x_2, \ldots, x_m) = d_1\left(x_{s_1}, d_2(x_{s_2}, \ldots, d_n(x_{s_n}, x_{s_n})\ldots)\right), \]
where \(1 \leqslant r_j \leqslant m, 1 \leqslant s_j \leqslant m\) \((j = 1, 2, \ldots, n)\), is unique. Let \(f_k\) and \(g_k\) \((k = 1, 2, \ldots, m)\) be the union in the sense of (43) of those operations \(d_j\) for which \(r_j = k\) and \(s_j = k\) respectively. For instance, if there is no index \(j\) for which \(r_j = k\), then \(f_k(x, y) = 0(x, y) = y\). Of course, we have the relations

\[
\bigcup_{k-1}^{m} f_k = \bigcup_{k-1}^{m} g_k = 1, \quad f_i \cap f_j = g_i \cap g_j = 0 \quad (i \neq j).
\]

Moreover, by (73) and (74),

\[
f(x_1, x_2, \ldots, x_m) = f_1(x_1, f_2(x_2, \ldots, f_m(x_m, x_m)) \ldots),
\]

\[
g(x_1, x_2, \ldots, x_m) = g_1(x_1, g_2(x_2, \ldots, g_m(x_m, x_m)) \ldots).
\]

Consequently, to prove the equation \(f = g\) it suffices to prove the equations \(f_k = g_k\) \((k = 1, 2, \ldots, m)\).

Put \(u_s^{(e)} = x\) and \(u_j^{(e)} = y\) if \(j \neq s\) \((s = 1, 2, \ldots, m)\). Since \(u_1^{(e)} = u_2^{(e)}\) if \(s = 3, 4, \ldots, m\) and \(u_1^{(2)} = u_3^{(2)}\), we have, by the hypothesis, the equations

\[
f(u_1^{(e)}, u_2^{(e)}, \ldots, u_m^{(e)}) = g(u_1^{(e)}, u_2^{(e)}, \ldots, u_m^{(e)}) \quad (s = 2, 3, \ldots, m).
\]

Applying Lemma 11 (formula 46) to disjoint operations \(f_1, f_2, \ldots, f_m\) we obtain the equation

\[
f(u_1^{(e)}, u_2^{(e)}, \ldots, u_m^{(e)})
\]

\[
= f_1(y, f_2(y, \ldots, f_{s-1}(y, f_s(x, f_{s+1}(y, \ldots, f_m(y, y)) \ldots)) \ldots))
\]

\[
= f_s(x, f_1(y, \ldots, f_{s-1}(y, f_{s+1}(y, \ldots, f_m(y, y)) \ldots)) \ldots) = f_s(x, y).
\]

Similarly we get the equation

\[
g(u_1^{(e)}, u_2^{(e)}, \ldots, u_m^{(e)}) = g_s(x, y).
\]

Thus, by (76), \(f_k = g_k\) for \(k = 2, 3, \ldots, m\). The equation \(f_1 = g_1\) is now a simple consequence of relations (75). The Lemma is thus proved.

**Lemma 18.** Let \(\mathcal{A}\) be a diagonal algebra, \(m \geq 4\) and \(f\) an \(m\)-ary not necessarily algebraic operation in \(\mathcal{A}\). If for each system \(h_1, h_2, \ldots, h_m\) of \((m-1)\)-ary algebraic operations in \(\mathcal{A}\) the composition

\[
f(h_1(x_1, x_2, \ldots, x_{m-1}), h_2(x_1, x_2, \ldots, x_{m-1}), \ldots, h_m(x_1, x_2, \ldots, x_{m-1}))
\]

is an algebraic operation in \(\mathcal{A}\) and \(f(x_1, x_2, \ldots, x_m) = x_1\) whenever at least two elements among \(x_1, x_2, \ldots, x_m\) are equal, then \(f(x_1, x_2, \ldots, x_m) = x_1\) whenever \(x_1, x_2, \ldots, x_m\) belong to a subalgebra of \(\mathcal{A}\) generated by less than \(m\) elements.
Proof. Let \( k \) be an integer satisfying the inequality \( 1 \leq k \leq m - 1 \) and let \( h_1, h_2, \ldots, h_m \) be \( k \)-ary algebraic operations. We shall prove the formula

\[
(77) \quad f(h_1(x_1, x_2, \ldots, x_k), h_2(x_1, x_2, \ldots, x_k), \ldots, h_m(x_1, x_2, \ldots, x_k)) = h_1(x_1, x_2, \ldots, x_k)
\]

by induction with respect to \( k \).

Since \( f(x, x, \ldots, x) = x \), formula (77) is obvious for \( k = 1 \). Consider the case \( k = 2 \). Put

\[
(78) \quad h_0(x_1, x_2, x_3) = f(h_1(x_1, x_2), h_2(x_1, x_3), h_3(x_1, x_3), h_4(x_1, x_2), \ldots, h_m(x_1, x_2)).
\]

The operation \( h_0 \) is algebraic. Moreover, by the hypothesis and the inequality \( m \geq 4 \), we have the equations

\[
\begin{align*}
  h_0(x_2, x_2, x_3) &= f(x_2, h_2(x_2, x_3), h_3(x_2, x_3), x_2, \ldots, x_3) = x_2 = h_1(x_2, x_2), \\
  h_0(x_3, x_2, x_3) &= f(h_1(x_3, x_2), x_3, x_3, h_4(x_3, x_2), \ldots, h_m(x_3, x_2)) = h_1(x_3, x_3).
\end{align*}
\]

Consequently, \( h_0(x_1, x_2, x_3) = h_1(x_1, x_2) \) whenever \( x_1 = x_2 \) or \( x_1 = x_3 \). Hence, by Lemma 17, we get the equation \( h_0(x_1, x_2, x_3) = h_1(x_1, x_2) \) for all elements \( x_1, x_2 \) and \( x_3 \). By (78) the equation \( h_0(x_1, x_2, x_2) = h_1(x_1, x_2) \) gives formula (77) for \( k = 2 \).

Suppose now that formula (77) holds for an integer \( k \) such that \( 2 \leq k \leq m - 2 \). Put

\[
(79) \quad g(x_1, x_2, \ldots, x_{k+1}) = f(h_1(x_1, x_2, \ldots, x_{k+1}), h_2(x_1, x_2, \ldots, x_{k+1}), \ldots, h_m(x_1, x_2, \ldots, x_{k+1})).
\]

By the inductive assumption we have the equations

\[
g(x_2, x_2, x_3, \ldots, x_{k+1}) = h_1(x_2, x_2, x_3, \ldots, x_{k+1})
\]

and

\[
g(x_3, x_2, x_3, x_4, \ldots, x_{k+1}) = h_1(x_3, x_2, x_3, x_4, \ldots, x_{k+1}).
\]

Hence, by Lemma 17, we get the equation

\[
g(x_1, x_2, \ldots, x_{k+1}) = h_1(x_1, x_2, \ldots, x_{k+1}),
\]

which, according to (79), implies formula (77). The Lemma is thus proved.

**Lemma 19.** If \( 2 \in \mathcal{P}(\mathbb{N}) \) and \( \mathcal{P}(\mathbb{N}) \neq \{2, 3, \ldots\} \), then either \( \mathcal{P}(\mathbb{N}) = \{s : 2 \leq s \leq n\} \), where \( n \geq 2 \) or \( \mathcal{P}(\mathbb{N}) = \{s : 2 \leq s \leq n\} \cup \{s : s \geq m\} \), where \( m > n \geq 2 \).
Moreover, } \mathcal{I}(\mathfrak{A}) = \{ s : 2 \leq s \leq n \}, \text{ where } n \geq 2 \text{ if and only if } \mathfrak{A} \text{ is an } n\text{-dimensional diagonal algebra. Further, } \mathcal{I}(\mathfrak{A}) = \{ s : 2 \leq s \leq n \} \cup \{ s : s \geq m \}, \text{ where } m > n \geq 2 \text{ if and only if } \mathfrak{A} = (A; \{d\} \cup F), \text{ where } (A; d) \text{ is an } n\text{-dimensional diagonal algebra, the class } F \text{ contains an } m\text{-ary operation depending on every variable, all operations } f \text{ from } F \text{ depend on at least } m \text{ variables and satisfy the equation } f(x_1, x_2, \ldots, x_k) = x_1 \text{ whenever the elements } x_1, x_2, \ldots, x_k \text{ belong to a subalgebra of the algebra } (A; d) \text{ generated by less than } m \text{ elements.}

\textbf{Proof.} \text{ It has been shown in Example 3 (Section II) that } \mathcal{I}(\mathfrak{A}) = \{ s : 2 \leq s \leq n \} \text{ for } n\text{-dimensional diagonal algebras } \mathfrak{A}. \text{ Suppose that } \mathfrak{A} = (A; \{d\} \cup F), \text{ where } (A; d) \text{ is an } n\text{-dimensional diagonal algebra, the class } F \text{ contains an } m\text{-ary operation depending on every variable, all operations } f \text{ from } F \text{ depend on at least } m \text{ variables and satisfy the equation } f(x_1, x_2, \ldots, x_k) = x_1 \text{ whenever } x_1, x_2, \ldots, x_k \text{ belong to a subalgebra of the algebra } (A; d) \text{ generated by less than } m \text{ elements. The equation}

\mathcal{I}(\mathfrak{A}) \cap \{2, 3, \ldots, m-1\} = \mathcal{I}((A; d)) \cap \{2, 3, \ldots, m-1\}

\text{is obvious. By Lemma 1 the set } \mathcal{I}(\mathfrak{A}) \text{ contains all integers } \geq m. \text{ Thus } \mathcal{I}(\mathfrak{A}) = \{ s : 2 \leq s \leq n \} \cup \{ s : s \geq m \}.

\text{Suppose that } 2 \in \mathcal{I}(\mathfrak{A}) \text{ and } \mathcal{I}(\mathfrak{A}) \neq \{2, 3, \ldots\}. \text{ By Lemma 14, } (A; A^{(2)}) \text{ is a diagonal algebra } (A; d). \text{ Since } 2 \in \mathcal{I}(\mathfrak{A}) \text{ and, consequently, } 2 \in \mathcal{I}((A; d)), \text{ the diagonal algebra } (A; d) \text{ is at least two-dimensional. Suppose that } \mathfrak{A} \neq (A; d). \text{ Then } \mathfrak{A} = (A; \{d\} \cup F_0), \text{ where the class } F_0 \text{ consists of all algebraic operations in the algebra } \mathfrak{A} \text{ which are not algebraic in the algebra } (A; d). \text{ By Lemma 13 each operation from } F_0 \text{ depends on at least four variables. Let } m \text{ be the integer such that the class } F_0 \text{ contains an } m\text{-ary operation depending on every variable and each operation from } F_0 \text{ depends on at least } m \text{ variables. Of course, } m \geq 4. \text{ By Lemmas 15, 16 and 18 there exists a subclass } F \text{ of the class } F_0 \text{ containing an } m\text{-ary operation, consisting of operations } f \text{ satisfying the equation } f(x_1, x_2, \ldots, x_k) = x_1 \text{ whenever the elements } x_1, x_2, \ldots, x_k \text{ belong to a subalgebra of } (A; d) \text{ generated by less than } m \text{ elements and, finally, satisfying the condition } (A; \{d\} \cup F_0) = (A; \{d\} \cup F). \text{ Hence and from the first part of the proof it follows that } \mathcal{I}(\mathfrak{A}) = \{ s : 2 \leq s \leq n \} \cup \{ s : s \geq m \}, \text{ where } n \text{ is the dimension of } (A; d). \text{ Since } \mathcal{I}(\mathfrak{A}) \neq \{2, 3, \ldots\}, \text{ we have the inequality } m > n \geq 2, \text{ which completes the proof of the Lemma.}

\textbf{Lemma 20.} \text{ If } 2 \notin \mathcal{I}(\mathfrak{A}), 3 \notin \mathcal{I}(A) \text{ and } \mathcal{I}(\mathfrak{A}) \neq 0, \text{ then } \mathcal{I}(\mathfrak{A}) = \{ s : s \geq m \}, \text{ where } m \geq 4. \text{ Moreover, } \mathcal{I}(\mathfrak{A}) = \{ s : s \geq m \}, \text{ where } m \geq 4 \text{ if and only if } \mathfrak{A} = (A; F), \text{ where the class } F \text{ contains an } m\text{-ary operation depending on every variable, all operations } f \text{ from } F \text{ depend on at least } m \text{ variables}.
and satisfy the equation \( f(x_1, x_2, \ldots, x_k) = x_1 \) whenever the system \( x_1, x_2, \ldots, x_k \) contains at most \( m - 1 \) different elements.

Proof. Suppose that \( 2 \notin \mathcal{I}(\mathcal{A}), \ 3 \notin \mathcal{I}(\mathcal{A}) \) and \( \mathcal{I}(\mathcal{A}) \neq \emptyset \). Let \( m \) be the integer such that the algebra \( \mathcal{A} \) contains an \( m \)-ary algebraic non-trivial operation depending on every variable and each non-trivial algebraic operation in \( \mathcal{A} \) depends on at least \( m \) variables. Of course, \( m \geq 4 \) and \( \mathcal{A} = (A; F_0) \), where \( F_0 \) denotes the class of all non-trivial algebraic operations in \( \mathcal{A} \). Applying Lemmas 15, 16 and 18 to one-dimensional diagonal algebras, i.e. to trivial algebras, we obtain the equation \( \mathcal{A} = (A; F) \), where \( F \) is a subset of \( F_0 \) containing an \( m \)-ary operation and consisting of operations \( f \) satisfying the equation \( f(x_1, x_2, \ldots, x_k) = x_1 \) whenever the elements \( x_1, x_2, \ldots, x_k \) belong to a subalgebra of a trivial algebra generated by less than \( m \) elements or, in other words, whenever the system \( x_1, x_2, \ldots, x_k \) contains at most \( m - 1 \) different elements. To prove the Lemma it suffices to prove the formula \( \mathcal{I}(\mathcal{A}) = \{s : s \geq m\} \). But this formula is a simple consequence of Lemma 1, which completes the proof.

Proof of Theorem 1. We consider four cases

\[(80) \quad 2 \in \mathcal{I}(\mathcal{A}), \quad 2 \notin \mathcal{I}(\mathcal{A}), \quad 3 \in \mathcal{I}(\mathcal{A}), \quad 4 \in \mathcal{I}(\mathcal{A}) ,\]

\[(81) \quad 2 \notin \mathcal{I}(\mathcal{A}), \quad 3 \in \mathcal{I}(\mathcal{A}), \quad 4 \in \mathcal{I}(\mathcal{A}) ,\]

\[(82) \quad 2 \notin \mathcal{I}(\mathcal{A}), \quad 3 \in \mathcal{I}(\mathcal{A}), \quad 4 \notin \mathcal{I}(\mathcal{A}),\]

\[(83) \quad 2 \notin \mathcal{I}(\mathcal{A}), \quad 3 \notin \mathcal{I}(\mathcal{A}).\]

In the case (80) we have, by Lemma 19 one of the following equations: \( \mathcal{I}(\mathcal{A}) = \{s : s \geq 2\} \), \( \mathcal{I}(\mathcal{A}) = \{s : 2 \leq s \leq n\} \), where \( n \geq 2 \) and \( \mathcal{I}(\mathcal{A}) = \{s : 2 \leq s \leq n\} \cup \{s : s \geq m\} \), where \( m > n \geq 2 \).

In the case (81) from Lemma 4 it follows that \( \mathcal{I}(\mathcal{A}) = \{s : s \geq 3\} \).

Further, in the case (82) we infer, according to Lemma 9, that either \( \mathcal{I}(\mathcal{A}) \) consists of all odd integers greater than 1 or \( \mathcal{I}(\mathcal{A}) \) consists of all odd integers greater than 1 and all integers \( \geq m \), where \( m \geq 5 \).

Finally, in the case (83), either the set \( \mathcal{I}(\mathcal{A}) \) is empty or, by Lemma 20, \( \mathcal{I}(\mathcal{A}) = \{s : s \geq m\} \), where \( m \geq 4 \). The Theorem is thus proved.

Proof of Theorem 2. The first and the second statements are a consequence of Lemma 19. The third and the fourth statements are a consequence of Lemma 9.
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INSTITUTE OF MATHEMATICS OF THE WROCLAW UNIVERSITY
INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES

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