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# ON PERIODIC EXTENSIONS OF FUNCTIONS

BY

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In this note I consider extensions of real functions defined on some unbounded set to continuous periodic functions. The results I present complete in a sense an answer (see [3]) I have given to a problem of Marczewski concerning such extensions.

As it is seen, an extension of a function is necessarily bounded and continuous. Therefore the same properties must be displayed by the functions to be extended. In the note [3] I have proved the existence of a sequence of points such that every bounded function defined on the set of terms of this sequence can be extended to a continuous periodic function. As the greatest lower bound of the set of distances between the terms of the sequence in question was positive, and, consequently, every function defined on the terms of this sequence was uniformly continuous, it was not necessary to assume explicitly the uniform continuity of the functions to be extend. Now I am going to prove a stronger theorem: there exists a sequence of *intervals* such that every uniformly continuous bounded function defined on the union of these intervals can be extended to a continuous periodic function. This result is in fact stronger than that presented in [3], because the left-hand end of the intervals we shall consider form sequences of the kind considered in [3]. An analogous result for almost periodic functions has been given by Hartman and Ryll-Nardzewski [2]. Owing to a result [5] (compare also [3] and [4]), they have proved the existence of an unbounded sequence of intervals of constant length such that every uniformly continuous bounded function defined on the union of these intervals can be extended to a continuous almost periodic function.

Thereafter I shall deal with the set of periods of all possible periodic extensions of a given function, and I shall prove that it has the power of the continuum under some conditions to be imposed on the sequence of intervals in question. This completes a result of Hartman [1], who has observed that such a set is always of measure zero.

THEOREM 1. *Let* $\delta_n > 0$, $\sum_{n=1}^{\infty} \delta_n = d < +\infty$, $\alpha_n > 0$, $\sum_{n=1}^{\infty} \alpha_n = a < +\infty$, $\mu > 0$, $\gamma = d + a + \mu$, $a_1 \geqslant \gamma$,

$$\frac{a_{n+1}}{a_n} \geqslant \frac{\gamma + \delta_{n+1}}{\delta_n}. \tag{1}$$

*Then every uniformly continuous bounded function* $\varphi(x)$ *defined on the set*

$$E = \bigcup_{n=1}^{\infty} \langle a_n, a_n + \alpha_n \rangle$$

*can be extended to a continuous periodic function defined on the whole real line.*

Proof. Denote by $m_n$ and $M_n$ the minimum and maximum of $\varphi(x)$ in the interval $\langle a_n, a_n + \alpha_n \rangle$ respectively, and put $c_n = 2^{-1}(m_n + M_n)$. Denote by $B$ the set of terms of the sequence $\{c_n\}$. Put $Y = \langle \inf c_n, \sup c_n \rangle$. The set $Y \setminus \bar{B}$ is open. If it is not empty, denote by $\Omega_n$ its components and associate with each component $\Omega_n = (\omega_n, \omega_n + |\Omega_n|)$ a positive number $\gamma_n$ in such a way that $\sum \gamma_n = 3^{-1} \cdot \mu$. Let

$$\mu \cdot 3^{-1} = \sum_{n=1}^{\infty} \beta_n,$$

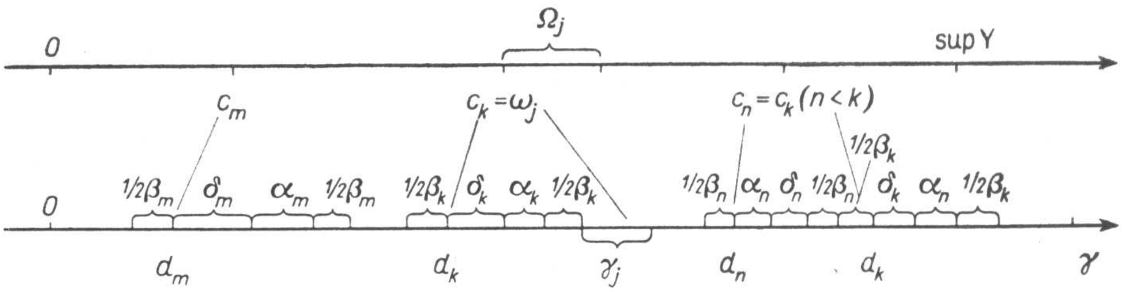


Fig. 1

where $\beta_n > 0$. Associate with every number $c_n$ the number (see fig. 1)

$$d_n = \sum_{c_i < c_n} (\delta_i + \alpha_i + \beta_i) + \sum_{\substack{c_i = c_n \\ i < n}} (\delta_i + \alpha_i + \beta_i) + \sum_{\omega_i < c_n} \gamma_i + 2^{-1} \cdot \beta_n. \tag{2}$$

The inequality $c_m < c_n$ implies $d_m < d_n$. From $c_n = c_k$ and $n < k$ results $d_n < d_k$. Let $\theta_n(r) = a_n / r$. Let $J_1$ be a closed interval such that $\theta_1(J_1) = \langle d_1, d_1 + \delta_1 \rangle$ (see fig. 2) and let $L_1 = \theta_2(J_1)$. We then have

$$J_1 = \langle a_1(d_1 + \delta_1)^{-1}, a_1 \delta_1^{-1} \rangle, \qquad L_1 = \langle a_2 a_1^{-1} d_1, a_2 a_1^{-1}(d_1 + \delta_1) \rangle.$$

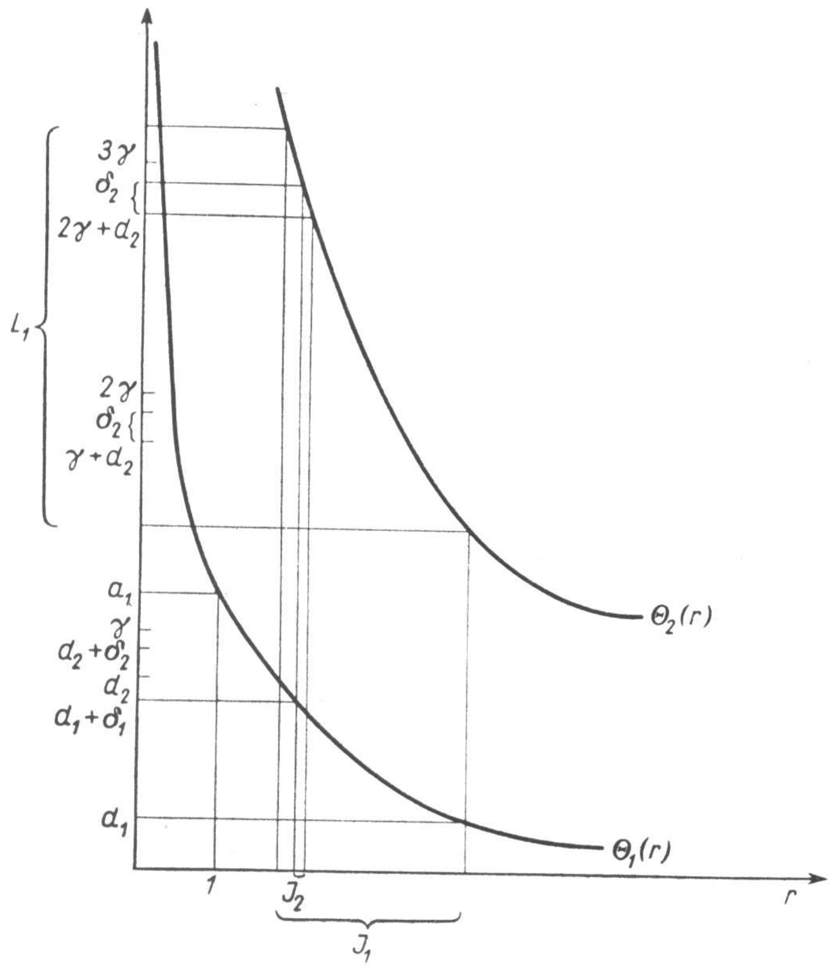


Fig. 2

Inequality (1) implies $|L_1| \geqslant \gamma+\delta_2$. Thus interval $L_1$ contains an interval of the form $\langle k_2\gamma+d_2, k_2\gamma+d_2+\delta_2\rangle$, where $k_2$ is an integer. Let us fix $k_2$ and denote by $J_2$ a closed interval such that $\theta_2(J_2) = \langle k_2\gamma+d_2, k_2\gamma+d_2+\delta_2\rangle$. Then $\theta_2(J_2) \subset L_1 = \theta_2(J_1)$. We then have $J_1 \supset J_2$. Suppose we have already defined for $i = 1, 2, \ldots, n-1$ the closed intervals $J_i$ and integers $k_i$ such that $J_{i-1} \supset J_i$ and $\theta_i(J_i) = \langle k_i\gamma+d_i, k_i\gamma+d_i+\delta_i\rangle$. Write $J_{n-1} = \langle r', r''\rangle$. We then have $r' = a_{n-1}(k_{n-1}\gamma+d_{n-1}+\delta_{n-1})^{-1}$ and $r'' = a_{n-1}(k_{n-1}\gamma+d_{n-1})^{-1}$. Let $L_{n-1} = \theta_n(J_{n-1})$. Then $L_{n-1} = \langle a_n r''^{-1}, a_n r'^{-1}\rangle = \langle a_n a_{n-1}^{-1}(k_{n-1}\gamma+d_{n-1}), a_n a_{n-1}^{-1}(k_{n-1}\gamma+d_{n-1}+\delta_{n-1})\rangle$. We thus have $|L_{n-1}| = a_n a_{n-1}^{-1}\delta_{n-1}$. In view of (1) we conclude

$$|L_{n-1}| \geqslant \gamma+\delta_n. \tag{3}$$

Interval $L_{n-1}$ contains at least one interval of the form $\langle k_n\gamma+d_n, k_n\gamma+d_n+\delta_n\rangle$, where $k_n$ is an integer. Let us fix $k_n$ and denote by $J_n$ a closed interval such that $\theta_n(J_n) = \langle k_n\gamma+d_n, k_n\gamma+d_n+\delta_n\rangle$. In this manner we define by induction a sequence $\{J_n\}$ of non-empty closed intervals. Obviously $J_{n-1} \supset J_n$. Thus there exists a point $r_0 \in \bigcap_{n=1}^{\infty} J_n$.

Because of the inequality $a_1 \geqslant \gamma$ and of the definition of $J_1$ we have $r_0 \geqslant 1$. For every $n$ we have

$$\theta_n(r_0) \in \langle k_n\gamma + d_n, k_n\gamma + d_n + \delta_n\rangle. \tag{4}$$

Write

$$F_n = \bigcup_{i=-\infty}^{+\infty} \langle a_n + ir_0\gamma, a_n + ir_0\gamma + \alpha_n\rangle.$$

We will prove that for $n \neq m$ we have

$$F_n \cap F_m = 0. \tag{5}$$

In fact, let $a_n = l_n r_0\gamma + r_n$, where $l_n$ is an integer and $0 \leqslant r_n < r_0\gamma$. Consequently $\theta_n(r_0) = l_n\gamma + r_n r_0^{-1}$. By (4) we have $k_n = l_n$ and $r_n r_0^{-1} \in \langle d_n, d_n + \delta_n\rangle$. Thus we see that the left-hand end of the interval $\langle a_n - l_n r_0\gamma, a_n - l_n r_0\gamma + \alpha_n\rangle$, which is a component of the set $F_n$, belongs to the interval $\langle r_0 d_n, r_0(d_n + \delta_n)\rangle$ and, because of $r_0 \geqslant 1$, the interval $\langle a_n - l_n\gamma, a_n - l_n\gamma + \alpha_n\rangle$ is contained in the interval $\langle r_0 d_n, r_0(d_n + \delta_n + \alpha_n)\rangle$. Because of (2) the last interval has no point in common with any interval $\langle r_0 d_m, r_0(d_m + \delta_m + \alpha_m)\rangle$ for $m \neq n$. For let $m \neq n$. Suppose we have $m < n$. We shall omit the discussion of the analógous case $n < m$. If $c_m = c_n$, then in view of (2) we have

$$d_n - d_m = \sum_{\substack{c_i = c_n \\ m \leqslant i < n}} (\delta_i + \alpha_i + \beta_i) + \tfrac{1}{2}\beta_n - \tfrac{1}{2}\beta_m$$

$$\geqslant \delta_m + \alpha_m + \tfrac{1}{2}\beta_m + \tfrac{1}{2}\beta_n > \delta_m + \alpha_m.$$

Therefore $d_m + \delta_m + \alpha_m < d_n$ and we conclude that the intervals in question are disjoint. If $c_m \neq c_n$, then we have either $c_m < c_n$ or $c_m < c_n$. We will consider but the first case, the second being analogous. In the first case we have

$$d_n - d_m = \sum_{c_m < c_i < c_n} (\delta_i + \alpha_i + \beta_i) +$$

$$+ \sum_{\substack{c_i = c_n \\ i < n}} (\delta_i + \alpha_i + \beta_i) + \sum_{\substack{c_i = c_m \\ i \geqslant m}} (\delta_i + \alpha_i + \beta_i) + \sum_{c_m \leqslant \omega_i < c_n} \gamma_i + \tfrac{1}{2}\beta_n - \tfrac{1}{2}\beta_m$$

$$> \delta_m + \alpha_m + \tfrac{1}{2}\beta_m > \delta_m + \alpha_m.$$

Hence similarly as in the case $c_m = c_n$ we conclude that the intervals in questions are disjoint. Consequently the intervals $\langle a_n - l_n\gamma, a_n - l_n\gamma + \alpha_n\rangle$ and $\langle a_m - l_m r_0\gamma, a_m - l_m r_0\gamma + \alpha_m\rangle$ are disjoint too, as they are contained in disjoint intervals. The components of the set $F_n$ whose left-hand ends belong to $\langle 0, r_0\gamma\rangle$ are thus pairwise disjoint. The remaining components result from those mentioned above through a translation by an integral multiplicity of $r_0\gamma$. Therefore (5) holds true.

Let $x \in \langle a_n + kr_0\gamma, a_n + kr_0\gamma + a_n\rangle$, where $k$ is an integer. Then $x - kr_0\gamma \in \langle a_n, a_n + \delta_n\rangle \subset E$. Define $f(x) = \varphi(x - kr_0\gamma)$. In this manner we get a periodic function defined on $\bigcup\limits_{n=1}^{\infty} F_n$ with a period $r_0\gamma$, which is an extension of $\varphi(x)$. We shall prove that it can be extended to a continuous periodic function defined on the whole real line. To this end it is sufficient to extend if from the set

$$D = \bigcup_{n=1}^{\infty} \langle a_n - l_n r_0\gamma, a_n - l_n r_0\gamma + a_n\rangle \subset \langle 0, \gamma\rangle$$

to a continuous function defined on $\langle 0, r_0\gamma\rangle$ with $f(0) = f(r_0\gamma)$.

Let $x_0$ be a limiting point of the set $D$. We shall prove that there exists the limit $\lim\limits_{x\to x_0} f(x)$. We shall distinguish two cases: 1) $x_0 \in D$, 2) $x_0 \in \bar{D}\setminus D$. In the first case there exists an $\bar{n}$ such that $x_0 \in \langle a_{\bar{n}} - l_{\bar{n}} r_0\gamma, a_{\bar{n}} - l_{\bar{n}} r_0\gamma + a_{\bar{n}}\rangle$. The last interval is contained in

$$\langle r_0 d_{\bar{n}}, r_0(d_{\bar{n}} + \delta_{\bar{n}} + a_{\bar{n}})\rangle \subset \big(r_0(d_{\bar{n}} - 2^{-1}\beta_{\bar{n}}), r_0(d_{\bar{n}} + \delta_{\bar{n}} + a_{\bar{n}} + 2^{-1}\beta_{\bar{n}})\big)$$

By (2), this last interval has no point in common with any interval $\langle r_0 d_m, r_0(d_m + \delta_m + a_m)\rangle$ for $m \neq n$ and, consequently, with intervals $\langle a_m - l_m r_0\gamma, a_m - l_m r_0\gamma + a_m\rangle$ contained in them. We thus have

$$\text{(6)} \qquad D \frown \big(r_0(d_{\bar{n}} - 2^{-1}\beta_{\bar{n}}), r_0(d_{\bar{n}} + \delta_{\bar{n}} + a_{\bar{n}} + 2^{-1}\beta_{\bar{n}})\big) = \langle a_{\bar{n}} - l_{\bar{n}} r_0\gamma, a_{\bar{n}} - l_{\bar{n}} r_0\gamma + a_{\bar{n}}\rangle = S.$$

The function $f(x)$ is continuous on $S$ and the limit

$$\lim_{\substack{x\to x_0\\ x\in S}} f(x) = f(x_0)$$

exists. In the second case we define a function $\psi(x)$ on $D$ by putting $\psi(x) = c_n$ for $x \in \langle a_n - l_n r_0\gamma, a_n - l_n r_0\gamma + a_n\rangle \subset \langle d_n, d_n + \delta_n + a_n\rangle$. As it is implied by (2), $\psi(x)$ is a non-decreasing function, thus it has at every two-sided limiting point of $D$ a left-hand and a right-hand limit, and at every one-sided limiting point of $D$ it has one of those limits. If at a limiting point there exist both limits, the left-hand one and the right-hand one, then they must be equal. For if we had $\psi(x-0) < \psi(x+0)$, then the interval $\big(\psi(x-0), \psi(x+0)\big)$ would be a component of $Y\setminus\bar{B}$, say the interval $(\omega_r, \omega_r + |\Omega_r|)$. As $x$ is both a left-hand and a right-hand limiting point of $D$ and $x \notin D$, there exist points $r_0 d_{n'}$, and $r_0 d_{n''}$ such that $r_0 d_{n'} < x < r_0 d_{n''}$, $r_0(d_{n''} - d_{n'}) < \gamma_r r_0$. This yields a contradition with definition (2), which implies

$$d_{n''} - d_{n'} = \sum_{c_{n'} < c_i < c_{n''}} (\delta_i + a_i + \beta_i) + \sum_{\substack{c_i = c_{n''} \\ i < n''}} (\delta_i + a_i + \beta_i) +$$

$$+ \sum_{\substack{c_i = c_{n'} \\ i \geqslant n'}} (\delta_i + a_i + \beta_i) + \tfrac{1}{2}\beta_{n''} - \tfrac{1}{2}\beta_{n'} + \sum_{c_{n'} \leqslant \omega_i < c_{n''}} \gamma_i > \gamma_r.$$

Therefore the function $\psi(x)$ has a limit at every point of $\bar{D} \setminus D$. For $x \in \bar{D} \setminus D$ define

$$f(x) = \lim_{\substack{t \to x \\ t \in D}} \psi(t).$$

We shall prove that $f(x)$ is continuous at the points of $\bar{D} \setminus D$. Let $x \in \bar{D} \setminus D$. Choose $\eta_1 > 0$ so that $|x - t| < \eta_1$ implied $|f(x) - \psi(t)| < \varepsilon/2$. Choose $\xi > 0$ so that $|t_1 - t_2| < \xi$ implied $|\varphi(t_1) - \varphi(t_2)| < \varepsilon/2$. This is possible because of the uniform continuity of $\varphi(x)$ on the set $E$. Let us have $a_n < \xi$ for $n > N$. Such an $N$ exists because the series $\sum_{n=1}^{\infty} a_n$ is convergent. Let $\eta_2 = \varrho(x, \bigcup_{i=1}^{N} \langle d_i, d_i + a_i \rangle)$. If $|x - t| < \eta_2$ and $t \in D$, then there exists an $i > N$ such that $t \in \langle d_i, d_i + a_i \rangle$. As $i > N$ we have $a_i < \xi$ and $|\psi(t) - f(t)| = |c_i - \varphi(k_i r_0 \gamma + t)| \leqslant M_i - m_i < \varepsilon/2$. We thus have $|f(x) - f(t)| \leqslant |f(x) - \psi(t)| + |\psi(t) - f(t)| < \varepsilon$ for $|x - t| < \min(\eta_1, \eta_2)$. As $\varepsilon$ is arbitrarily small, this implies the continuity of the function $f(x)$ at $x$, if considered on the set $\bar{D}$.

In view of the definition (2) of the numbers $d_n$ we have $0 \leqslant \inf D$, $\varrho(D, \gamma) > 3^{-1}\varepsilon$, and the more so $\varrho(D, r_0\gamma) > 3^{-1}\varepsilon > 0$. Put $f(r_0\gamma) = f(0)$ and if $0 \notin \bar{D}$ define $f(0)$ arbitrarily. In this way the function $f(x)$ is extended over a closed set $\bar{D} \cup \{r_0\gamma\} \cup \{0\} \subset \langle 0, r_0\gamma \rangle$ with the preservation of continuity. As it is known, it can now be extended continuously over the whole interval $\langle 0, r_0\gamma \rangle$, which completes the proof.

THEOREM 2. *Under assumptions of theorem* 1 *with inequality* (1) *replaced by the inequality*

$$\frac{a_{n+1}}{a_n} \geqslant \frac{2\gamma + \delta_{n+1}}{\delta_n} \tag{7}$$

*every bounded function $\varphi(x)$ defined and uniformly continuous over $E$ can be extended in infinitely many ways to a continuous periodic functions defined over the whole real line and the set of the periods of all its extensions has the power of the continuum.*

Proof. If inequality (7) holds true, then so does inequality (1). Thus in view of theorem 1 there exists an extension of $\varphi(x)$ to a continuous periodic function with a period equal to $r_0\gamma$, where $r_0$ and $\gamma$ are defined as in the proof of theorem 1. The number $r_0$ is not uniquely defined. We shall prove that under assumption (7) the set of the numbers $r_0$ has the power of the continuum.

In fact, in the proof of theorem 1 inequality (3) was deduced from (1). Now instead of (3) we deduce from (7) the inequality

$$|L_{n-1}| \geqslant 2\gamma + \delta_n.$$

Therefore the interval $L_{n-1}$ contains at least two intervals of the form $\langle k_n\gamma + d_n, k_n\gamma + d_n + \delta_n\rangle$, where $k_n$ is an integer. With them there are associated two different intervals $J_n$. Denote them by $J_{n,0}$ and $J_{n,1}$. We have

$$J_{n,0} \frown J_{n,1} = 0.$$

When defining the intervals $J_n$ we have to choose one of the intervals $J_{n,i}$ by letting $i = 0$ or $i = 1$. The number $r_0$ is defined by a decreasing sequences of intervals $J_{n,i_n}$ with $i_n = 0$ or 1. There are as many such sequences as there are sequences of 0's and 1's, so they form a set of the power of the continuum. It is implied by (8) that different sequences $\{i_n\}$ determine different numbers $r_0$. This completes the proof.

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