SET-MEMBERSHIP IDENTIFIABILITY OF NONLINEAR MODELS AND RELATED PARAMETER ESTIMATION PROPERTIES

CARINE JAUBERTHIE\textsuperscript{a}, LOUISE TRAVÉ-MASSUYÈS\textsuperscript{a,*}, NATHALIE VERDIÈRE\textsuperscript{b}

\textsuperscript{a}LAAS-CNRS
University of Toulouse, UPS, 7 avenue du Colonel Roche, 31400 Toulouse, France
email: \{cjaubert,louise\}@laas.fr

\textsuperscript{b}UNIHAVRE, LMAH
Normandy University, FR-CNRS-3335, ISCN, 76600 Le Havre, France
email: verdiern@univ-lehavre.fr

Identifiability guarantees that the mathematical model of a dynamic system is well defined in the sense that it maps unambiguously its parameters to the output trajectories. This paper casts identifiability in a set-membership (SM) framework and relates recently introduced properties, namely, SM-identifiability, $\mu$-SM-identifiability, and $\varepsilon$-SM-identifiability, to the properties of parameter estimation problems. Soundness and $\varepsilon$-consistency are proposed to characterize these problems and the solution returned by the algorithm used to solve them. This paper also contributes by carefully motivating and comparing SM-identifiability, $\mu$-SM-identifiability and $\varepsilon$-SM-identifiability with related properties found in the literature, and by providing a method based on differential algebra to check these properties.

Keywords: identifiability, bounded uncertainty, set-membership estimation, nonlinear dynamic models.

1. Introduction

Identifiability is an important concept that decides to what extent the parameter values of a mathematical model can be uniquely inferred from input-output measurements, assuming that the model has the same structure as the system (Nelles, 2002). Mathematically, this means that there exists an unambiguous mapping between the model parameters and the output trajectories. Identifiability is hence a pre-condition for safely running a parameter estimation algorithm and obtaining reliable results.

In the last years, there has been quite a lot of emphasis on bounded-error models, as opposed to stochastic ones, for achieving several tasks, e.g., fault diagnosis and fault tolerant control (Puig, 2010; Seybold et al., 2015), robust robot localization (Kieffer et al., 2011), reachability analysis (Auer et al., 2013; Maiga et al., 2016). This has been stressed by the success of operational estimation methods aiming at computing sets guaranteed to contain the feasible parameter/state set, i.e., the set of all the parameter/state vectors consistent with the specified bounds. This is why bounded, or also called \textit{set-membership}, estimation is qualified as \textit{guaranteed} (Kieffer et al., 2002). In this paper, we use the term \textit{set-membership}, abbreviated as SM, having in mind that the type of sets can be of different kinds, such as ellipsoids (Kurzhanski and Valyi, 1997), boxes (Kieffer and Walter, 2011), parallelotopes (Chiscii et al., 1996), zonotopes (Alamo et al., 2005), or other polytopes.

Interval analysis has brought a set of tools that indifferently apply to linear and nonlinear systems (Jaulin et al., 2001) as opposed to ellipsoidal and zonotope-based estimation methods. Furthermore, its efficiency has been considerably enhanced by recent constraint propagation techniques (Chabert and Jaulin, 2009; Kieffer and Walter, 2011; Maiga et al., 2013), resulting in the most appropriate paradigm to deal with nonlinearities. Interval analysis has brought a set of tools that indifferently apply to linear and nonlinear systems (Jaulin et al., 2001) as opposed to ellipsoidal and zonotope-based estimation methods. Furthermore, its efficiency has been considerably enhanced by recent constraint propagation techniques (Chabert and Jaulin, 2009; Kieffer and Walter, 2011; Maiga et al., 2013), resulting in the most appropriate paradigm to deal with nonlinearities.

Identifiability of SM nonlinear models has been shown to give rise to three concepts: SM-identifiability, $\mu$-SM-identifiability, and $\varepsilon$-SM-identifiability that we introduced earlier (Jauberthie et al., 2011; 2013). In this paper, we are interested in the way these properties impact the SM parameter estimation (SM-PE) problem. This
issue is characterized by two new properties. Soundness guarantees that the feasible parameter set (FPS) is reduced to one single bounded connected set. On the other hand, \( \varepsilon \)-consistency is a numerical property that guarantees that the FPS and the solution set returned by a parameter estimation algorithm with precision \( \varepsilon \) are composed of an equal number of mutually disjoint connected sets. While abundant literature exists on SM-PE (Jaulin et al., 2001; Raïssi et al., 2004; Kieffer and Walter, 2011; Milanese et al., 2013; Herrero et al., 2016), these problems have never been discussed in relation to SM-identifiability.

The paper is organized as follows. Section 2 recalls the definitions of SM-identifiability, \( \mu \)-SM-identifiability, and \( \varepsilon \)-SM-identifiability, and a method for checking these properties. Section 3 brings a first contribution with a thorough analysis of the links between these ones and related properties existing in the literature. Section 4 introduces the properties of SM-PE problems, namely, soundness and \( \varepsilon \)-consistency, as Sections 2 and 6 derive the conditions that guarantee these properties. Finally, Section 7 concludes the paper and discusses perspectives.

2. Set-membership identifiability

This section resumes the framework proposed by Jauberthie et al. (2013) for SM-identifiability for the class of systems formalized below.

2.1. Class of systems. The models considered in this paper are bounded-error uncertain nonlinear models, controlled or uncontrolled, of the following form:

\[
\Gamma = \begin{cases} 
\dot{x}(t,p) = f(x(t,p), u(t), p), \\
y(t,p) = h(x(t,p), p), \\
x(t_0, p) = x_0 \in X_0, \\
p \in \mathcal{P} \subseteq \mathcal{U}_\mathcal{P}, 
\end{cases} 
\tag{1}
\]

where

- \( x(t,p) \in \mathbb{R}^n \) and \( y(t,p) \in \mathbb{R}^m \) denote the state variables and the outputs at time, \( t \) respectively;
- \( u(t) \in \mathbb{R}^p \) is the input vector at time \( t \); in the case of uncontrolled models, \( u(t) \) is equal to 0;
- the initial conditions \( x_0 \), if any, are assumed to belong to a bounded set \( X_0 \), and one assumes that \( X_0 \) does not contain equilibrium points of the system;
- the parameter vector \( p \) belongs to a connected set \( P \) assumed to be included in \( \mathcal{U}_\mathcal{P} \), where \( \mathcal{U}_\mathcal{P} \subseteq \mathbb{R}^p \) is an a priori known set of admissible parameters; the components of \( p \) are denoted by \( p_i \);
- the functions \( f \) and \( h \) are real and analytic on \( M \), where \( M \) is an open set of \( \mathbb{R}^n \) such that \( x(t,p) \in M \) for every \( t \in [t_0, T] \) and \( p \in P \), \( T \) is a finite or infinite time bound.

In the following, \( Y(P) \) denotes the set of output trajectories, the solution of \( \Gamma \) for any \( p \in P \), and is also called the output of \( \Gamma \) arising from \( P \). \( P^c \) denotes the complement of \( P \) in \( \mathcal{U}_\mathcal{P} \).

2.2. Useful concepts. Let us consider a nonempty connected set \( \Pi \subseteq \mathbb{R}^p \) and \( d \) a usual norm on \( \mathbb{R}^p \) and its associated distance.

The distance \( d \) between two sets \( \Pi_1 \) and \( \Pi_2 \) of \( \mathbb{R}^p \) is defined by

\[
d(\Pi_1, \Pi_2) = \min_{\pi_1 \in \Pi_1, \pi_2 \in \Pi_2} d(\pi_1, \pi_2).
\]

Let us define \( \delta(\Pi) \) as the diameter of \( \Pi \). \( \delta(\Pi) \) is given by the least upper bound of \( \{d(\pi_1, \pi_2) \mid \pi_1, \pi_2 \in \Pi \} \).

If \( \Pi \) is not bounded, we define \( \delta(\Pi) = +\infty \) (Bourbaki, 1989). On the metric space \( (\Pi, d) \), let \( \mu \) be a continuous map from \( \Pi \) to \( \mathbb{R} \).

As an extension of the definition of contraction by Munkres (1975), we define \( \mu \) as a set contraction if there is a nonnegative number \( k < 1 \) such that for all \( \Pi_1, \Pi_2 \subseteq \mathbb{R}^p \), \( d(\mu(\Pi_1), \mu(\Pi_2)) < kd(\Pi_1, \Pi_2) \).

In the following, \( \| \cdot \| \) denotes the Euclidean norm, \( \| \cdot \|_\infty \) the maximum norm, and \( \| \cdot \|_1 \) the norm 1. These may be defined on \( \mathbb{R}^n \), where \( \alpha \in \{n, p, m\} \), depending on the case.

2.3. Definitions. The proposed definitions are given for controlled systems but they can be formulated in a similar manner for uncontrolled ones assuming that \( u(t) = 0 \).

Definition 1. Given the model \( \Gamma \) expressed by (1), consider a nonempty connected set \( P^* \subseteq \mathcal{U}_\mathcal{P} \) and another set \( \bar{P} \subseteq \mathcal{U}_\mathcal{P} \). Then \( P^* \) is globally SM-identifiable if there exists an input \( u \) such that \( Y(P^*) \neq \emptyset \) and \( Y(P^*) \cap Y(\bar{P}) \neq \emptyset \).

Definition 1 states that a connected set \( P^* \) is globally SM-identifiable if the output of \( \Gamma \) arising from \( P^* \) does not share any trajectory with the output of \( \Gamma \) arising from any set \( P \subseteq P^c \). As an example, consider the following nonlinear system of the form (1):

\[
\dot{x} = x + t \cos(p), \quad x(t_0) = x_0, 
\tag{2}
\]

\( ^{1} \)The assumption that \( f \) and \( h \) are analytic on \( M \), and hence infinitely differentiable, is needed in Section 6 for the use of differential algebra.

\( ^{2} \)To keep the concept intuitive, it is a deliberate abuse of language to call \( d(\Pi_1, \Pi_2) \) a distance between the two sets \( \Pi_1 \) and \( \Pi_2 \) of \( \mathbb{R}^p \), even though it does not satisfy all the assumptions of a distance, in particular the triangular inequality.
where $p$ is a bounded-error parameter for which the admissible set is $\mathcal{U}_P = [0, 2\pi]$. The solution of (3) is $x(t) = x_0e^t + (-1 - t + e^t)\cos(p)$. It is clear that this system is not globally identifiable. It is enough to notice that any pair $(p_1 = \pi - \alpha, p_2 = \pi + \alpha)$, with $\alpha \in [0, \pi]$, results in the same trajectory, since $\cos(\pi - \alpha) = \cos(\pi + \alpha)$ for $\alpha \in [0, \pi]$. However, the trajectories arising from any set $P^* = [\pi - \alpha, \pi + \alpha]$ with $\alpha \in [0, \pi]$ are different from any trajectory arising from other regions of the parameter space. $P^*$ is then said to be globally SM-identifiable.

The definition of $\mu$-SM-identifiability has been proposed to ensure that the set $P^*$ may be contracted as small as desired while still retaining the SM-identifiability property. For this purpose, a contraction $\mu$ is applied to $P^*$ and, by the Banach fixed-point theorem, it implies that the diameter of $\mu(P^*)$ tends to zero (Munkres, 1975).

**Definition 2.** A nonempty connected set $P^* \subseteq \mathcal{U}_P$ is globally $\mu$-SM-identifiable if $\mu(P^*)$ is globally SM-identifiable for any contraction $\mu$ from $P^*$ to $P^*$.

This implies the following proposition.

**Proposition 1.** If the nonempty connected set $P^* \subseteq \mathcal{U}_P$ is globally $\mu$-SM-identifiable, then it is globally SM-identifiable. The converse is not true.

**Proof.** For the converse, consider the system (3) and the set $P^* = [\pi - \alpha, \pi + \alpha]$ with $\alpha \in [0, \pi]$ as before. $P^*$ has been shown to be globally SM-identifiable but it is not $\mu$-SM-identifiable since, assuming $\alpha_1, \alpha_2 \in [0, \pi], \alpha_1 \geq \alpha_2$, any set $P^*_i^1 = [\pi - \alpha_1, \pi - \alpha_2] \subseteq P^*$ shares trajectories with its complementary set $P^*_i^2$ that contains $[\pi + \alpha_2, \pi + \alpha_1]$. $\square$

If the diameter of $\mu(P^*)$, $\delta(\mu(P^*))$, cannot be lower than $\varepsilon$ without loosing SM-identifiability, we refer to $\varepsilon$-SM-identifiability (Jauberthie et al., 2013).

**Definition 3.** Consider an SM-identifiable nonempty connected set $P^* \subseteq \mathcal{U}_P$. Then $P^*$ is globally $\varepsilon$-SM-identifiable if there exists a set contraction $\mu$ from $P^*$ to $P^*$ such that $\delta(\mu(P^*)) = \varepsilon$ and $\mu(P^*)$ is globally SM-identifiable, and for all $\tilde{\mu}$ such that $\tilde{\mu}(P^*) \subset \mu(P^*)$, $\tilde{\mu}(P^*)$ is not globally SM-identifiable.

To summarize, interpreting identifiability in the SM framework leads to two definitions, depending on whether one considers a set as a whole (SM-identifiability) or also cares about the properties of its proper subsets ($\mu$-SM-identifiability). $\mu$-SM-identifiability can be seen as subsuming classical identifiability in the sense that if $P^*$ is $\mu$-SM-identifiable, it implies that any $p \in P^*$ is identifiable in the classical sense (Ljung and Glad, 1994). $\varepsilon$-SM-identifiability is a kind of structural $\mu$-SM-identifiability since subsets of a delimited diameter $\varepsilon$ that are SM-identifiable although not $\mu$-SM-identifiable are accepted. The reader is referred to the work of Jauberthie et al. (2011) for the extension to structural and local counterparts of these properties.

### 3. SM-identifiability and related concepts

The links between ($\mu$)-SM-identifiability and classical and interval identifiability were provided by Jauberthie et al. (2011; 2013). In this section, we are interested in the links with $\varepsilon$-global identifiability (Jauberthie et al., 2001) and partial injectivity (Lagrange et al., 2008). These links allow us to propose a method for checking ($\mu$)-SM-identifiability.

#### 3.1. Links with $\varepsilon$-global identifiability.

Global identifiability in $P^* \subset \mathcal{U}_P$ (g.i.i.$P^*$) was proposed by Braems et al. (2001) as a means to provide a stronger conclusion than structural identifiability, guaranteeing that atypical regions of nonidentifiability do not exist in the parameter space.

**Definition 4.** Given $(u, x_0) \in \mathbb{R}^r \times X_0$, the parameter $p_i$ is globally identifiable in $P^*$ (g.i.i.$P^*$) if

$$\forall (p, \bar{p}) \in P^*, y(\cdot, p) \equiv y(\cdot, \bar{p}) \Rightarrow p_i = \bar{p}_i.$$

The originality of Braems et al. (2001) is to propose a practical way to formulate the condition of Definition 4 which consists in checking the condition

$$\exists (p, \bar{p}) \in P^* \times P^*$$

such that $y(\cdot, p) \equiv y(\cdot, \bar{p}), \|p - \bar{p}\|_{\infty} > 0$. (4)

This is a constraint satisfaction problem (CSP) that can be solved in a guaranteed way by interval constraint propagation (ICP). In practice, Braems et al. (2001) state that checking the condition (4) comes back to checking

$$\exists (p, \bar{p}) \in P^* \times P^*$$

such that $y(\cdot, p) \equiv y(\cdot, \bar{p}), \|p - \bar{p}\|_{\infty} > \varepsilon$, (5)

which is defined as $\varepsilon$-g.i.i.$P^*$. We have the following results.

**Proposition 2.** $P^*$ is globally $\mu$-SM-identifiable with respect to $P^*$ (in the sense that $\mathcal{U}_P$ is reduced to $P^*$) if and only if (4) is satisfied.

**Proof.** Jauberthie et al. (2013) provided the proof that if $P^*$ is globally $\mu$-SM-identifiable, then equivalently any $p \in P^*$ is globally identifiable with respect to $P^*$, hence satisfying the conditions (3) and (4). $\square$

**Proposition 3.** If $P^*$ is globally $\varepsilon$-SM-identifiable with respect to $P^*$, then the condition (5) is satisfied.
Proof. $P^*$ is globally $\varepsilon$-SM-identifiable (cf. Definition 3) with respect to $P^*$ if and only if there exists some subset $P \subseteq P^*$ such that $\delta(\hat{P}) = \varepsilon$ and the interior of $P$, denoted by $\text{int}(P)$, as well as any $P' \subseteq P$ is not globally SM-identifiable, hence not globally identifiable. In such a case, for all $p, \bar{p} \in P^*$, $\text{int}(\hat{P})$ satisfies the condition (5). The converse is not true because, when the condition (5) is satisfied, it does not provide any information about subsets $P' \subseteq \hat{P}$ such that $\delta(\hat{P}) \leq \varepsilon$.

From the above propositions, the condition (5) does not allow one to decide between $\mu$-SM-identifiability and $\varepsilon$-SM-identifiability. It can be considered to check $\mu$-SM-identifiability accepting a numerical precision of $\varepsilon$.

### 3.2. Links with partial injectivity

The definition of partial injectivity of a function was introduced in Lagrange et al. (2008). This notion perfectly characterizes $\mu$-SM-identifiability. A second definition named restricted-partial injectivity is proposed in this paper in order to characterize global SM-identifiability.

**Definition 5.** Consider a function $f : A \rightarrow B$ and any set $A_1 \subseteq A$. The function $f$ is said to be a partial injection of $A_1$ over $A$, or $(A_1, A)$-injective, if $\forall a_1 \in A_1, \forall a \in A$, $a_1 \neq a \Rightarrow f(a_1) \neq f(a)$. $f$ is said to be $A$-injective if it is $(A, A)$-injective.

In Lagrange et al. (2008), an algorithm based on interval analysis for testing the injectivity of a given differentiable function is presented and a solver called IAVIA (injectivity analysis using interval analysis) implemented in C++ is mentioned. For a given function, the solver partitions a given box in two domains: a domain on which the function is partially injective and an indeterminate domain on which the function may or may not be injective.

In order to characterize global SM-identifiability, the notion of restricted-partial injectivity is introduced.

**Definition 6.** Consider a function $f : A \rightarrow B$ and any set $A_1 \subseteq A$. The function $f$ is said to be a restricted-partial injection of $A_1$ over $A$, or $(A_1, A)$-R-injective, if $\forall a_1 \in A_1, \forall a \in A$, $f(a_1) \neq f(a)$.

In the following proposition, partial injectivity and restricted partial injectivity are interpreted in terms of trajectories, and this formulation makes the direct link with the definition of SM-identifiability and $\mu$-SM-identifiability possible.

Consider the set of outputs $S_u$ arising from $U_P$ for a given input $u$.

**Proposition 4.** Given the model $\Gamma$, $P^*$ is globally SM-identifiable (resp. $\mu$-SM-identifiable) for an input $u$ if and only if the function $\phi : U_P \rightarrow S_u : p \rightarrow y(\cdot, p)$ is $(P^*, U_P)$-injective (resp. $(P^*, U_P)$-R-injective).

**Proof.**

(Necessity) From the definition of global SM-identifiability, $P^*$ and its complementary do not share trajectories; hence, there do not exist common trajectories arising from these two sets, which implies that $\phi$ is $(P^*, U_P)$-R-injective.

If $P^*$ is $\mu$-SM-identifiable, then the property of global SM-identifiability is satisfied for any $\mu(P^*)$, $\mu$ being a contraction from $P^*$ to $P^*$, which implies that for any $\hat{P}$ included in the complement of $\mu(P^*)$, $Y(\mu(P^*))$ and $Y(\hat{P})$ have no common trajectories. In other words, from the Banach fixed-point theorem, the trajectory arising from $p \in P^*$ is different from any trajectory arising from $U_P \setminus \{p\}$ and hence $\phi$ is $(P^*, U_P)$-injective.

(Sufficiency) If $\hat{P}$ is such that $P^* \cap \hat{P} = \emptyset$, $\hat{P}$ is included in the complementary of $P^*$, and since $\phi$ is $(P^*, U_P)$-R-injective, there exist no common trajectories arising from these two sets; hence $P^*$ is globally SM-identifiable.

Assume now that $\phi$ is $(P^*, U_P)$-injective and that, for a contraction $\mu$, $Y(\mu(P^*))$ and $Y(\hat{P})$ have common trajectories. Then these trajectories arise from the same parameter. This implies that $\mu(P^*)$ and $\hat{P}$ have a nonempty intersection and that $P^*$ is globally $\mu$-SM-identifiable.

**Corollary 1.** The following properties are equivalent:

- $P^*$ is globally $\mu$-SM-identifiable;
- the function $\phi : U_P \rightarrow S_u : p \rightarrow y(\cdot, p)$ is $(P^*, U_P)$-injective;
- the condition (4) is satisfied.

**Proof.** The proof directly comes from Propositions 2 and 4.

**Corollary 2.** $P^*$ is globally $\varepsilon$-SM-identifiable implies that $\phi$ is $(P, U_P)$-R-injective, with $\hat{P} \subseteq P^*$ and $\delta(\hat{P}) \geq \varepsilon$. The converse is not true.

**Proof.** The necessity part of the proof of Proposition 4 applies and the inverse is not true for the same reasons as in the proof of Proposition 1.

Testing $(P^*, U_P)$-injectivity or $(P^*, U_P)$-R-injectivity numerically can be done with an adaptation of IAVIA (Lagrange et al., 2008) but it does not allow us to decide between $\mu$-SM-identifiability and $\varepsilon$-SM-identifiability or SM-identifiability and $\varepsilon$-SM-identifiability.
Set-membership identifiability of nonlinear models and related parameter estimation properties

3.3. Differential algebra method to perform SM-identifiability analysis. Proposition 4 points at an operational method to check SM and μ-SM-identifiability provided that the function \( \varphi : \mathcal{U}_R \rightarrow \mathcal{S}_u : p \rightarrow y_i(t, p) \) that maps parameters and trajectories is known. Differential algebra (Kolchin, 1973) is an efficient tool. The Rosenfeld–Groebner algorithm, which is an elimination algorithm (Boulier, 1994) implemented in the package DifferentialAlgebra of Maple, is an efficient tool. The algorithm is based on the use of relations linking outputs, inputs and parameters of the model. These relations are more precisely differential polynomials whose indeterminates are the variables \( y \) and \( u \), and coefficients are rational expressions in \( p \).

For obtaining such polynomials, the Rosenfeld–Groebner algorithm, which is an elimination algorithm (Boulier, 1994) implemented in the package DifferentialAlgebra of Maple, is an efficient tool. The Rosenfeld–Groebner algorithm is used to eliminate state variables—to obtain the relations linking only outputs, inputs, and parameters. With the elimination order \( \{p\} < \{y, u\} < \{x\} \) (Kolchin, 1973; Denis-Vidal et al., 2001b), several solutions are delivered by the algorithm. One is called the characteristic presentation because it corresponds to the general solution, the others being particular solutions.

The characteristic presentation contains differential polynomials linking outputs, inputs, and parameters of the form

\[
R_i(y, u, p) = m_{0i}(y, u) + \sum_{k=1}^{n_i} \theta_{ki}(p)m_{ki}(y, u), \quad i = 1, \ldots, m,
\]

where \( \theta_{ki}(p) \) are rational in \( p \), \( \theta_{ki} \neq \theta_{k \neq i} \) (\( u \neq v \)), \( m_{ki}(y, u) \) are differential polynomials with respect to \( y \), \( u \) and \( m_{0i}(y, u) \neq 0 \). \( \{\theta_{ki}(p)\}_{1 \leq k \leq n_i} \) is called the exhaustive summary of \( R_i \).

The size of the system is the number of outputs. For the time being, we assume that \( i = 1 \); that is, there is one output and \( n_1 = n, R_1 = R, m_{1k}(y, u) = m_k(y, u) \). The case of more outputs is considered at the end of this section.

Consider \( t_{n}^1 \) the right limit of \( t_{n-1} \) and \( l \) the higher order derivative of \( y \) in (6). \( \Delta R(y, u) \) denotes the functional determinant formed from \( \{m_k(y, u)\}_{1 \leq k \leq n} \) and given by the Wronskian (Denis-Vidal et al., 2001a)

\[
\Delta R(y, u) = \begin{vmatrix}
    m_1(y, u) & \cdots & m_n(y, u) \\
    m_1(y, u)^{(1)} & \cdots & m_n(y, u)^{(1)} \\
    \vdots & \ddots & \vdots \\
    m_1(y, u)^{(n-1)} & \cdots & m_n(y, u)^{(n-1)} \\
\end{vmatrix}.
\]

Theorem 1. (Jauberthie et al., 2011) Assume that the functional determinant \( \Delta R(y, u) \) is not identically equal to zero. Consider \( P^* \) a connected subset of \( \mathcal{U}_R \).

If the function defined by \( \phi : p \mapsto (p_1, \ldots, p_p) \Rightarrow (\theta_1(p), \ldots, \theta_n(p), y(t_0^1, p), \ldots, y^{(n-1)}(t_0^1, p)) \) is \( (P^*, \mathcal{U}_R) \)-injective (resp. \( (P^*, \mathcal{U}_R^1) \)-injective) then \( P^* \) is globally SM-identifiable (resp. \( \mu \)-SM-identifiable).

Furthermore, if for a contraction \( \mu, \mu(P^*) \) has a diameter equal to \( \varepsilon \) and \( \phi \) is \( (\mu(P^*), \mathcal{U}_R) \)-injective but not \( (\mu(P^*), \mathcal{U}_R^1) \)-injective then \( P^* \) is \( \varepsilon \)-SM-identifiable. In the two cases, if the coefficient of \( y^{(1)} \) in (6) is not equal to 0 at \( t_0 \), then the converse is valid.

Remark 1. If \( m \geq 1 \), then for each of the obtained \( m \) differential polynomials \( R_i(y, u, p) \), the functional determinant is evaluated. If it is not identically equal to zero, the associated exhaustive summary is added to the image of the function \( \phi \) for which (partial) injectivity has to be studied.

Theorem 1 has been used by Ravanbod et al. (2014) to provide an operational method for analyzing identifiability in an SM framework. First, the \( \mu \)-SM-identifiable parameter subsets are determined with IAVIA. Then, determining the maxima and minima of the function \( \phi \) allows one to assess SM-identifiable subsets and subsets that are neither SM nor \( \mu \)-SM-identifiable.

4. SM parameter estimation and properties

In this section, the SM-PE problem is presented and two important properties are introduced, namely, soundness and \( \varepsilon \)-consistency. SM-identifiability is shown to play a key role in relation to this problem.

Classical parameter estimation considers a time series of noisy measured output data \( y_m(t_i), i = 0, \ldots, h \), where \( y_m(t) \in \mathbb{R}^m \), generated by the real system on the interval \([0, T]\). The problem is formulated as finding the parameter vector \( p^* \) for which the outputs produced by the

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4Another technique based on the power series expansion method inspired by Pohjanpalo (1978) was also proposed by Jauberthie et al. (2011).

5When initial conditions are not considered, the function \( \phi \) becomes \( \phi : p \mapsto (p_1, \ldots, p_p) \Rightarrow (\theta_1(p), \ldots, \theta_n(p)) \) and the converse of the theorem is not valid.

6This assumption consists in verifying the linear independence of the \( m_k(y, u), k = 1, \ldots, n \). To this end, it is sufficient to find a time point at which the Wronskian is nonzero. In the framework of differential algebra, this condition consists in verifying that this functional determinant is not ideally obtained after eliminating state variables. In practice, it can be checked with the function BelongTo of the package Differential Algebra of Maple 16.
model best match the measured data according to some criterion. Minimal least squares is a common method, which is formulated as

\[ p^* = \arg \min_{p \in \mathbb{R}^p} \sum_{t=t_0}^{t_h} \| y_m(t) - y(t, p) \|^2. \]

The SM-PE problem assumes that the measured outputs are corrupted by bounded-error terms that may originate from the system parameters varying within specified bounds, bounded noise, or sensor precision such that \( y_m(t_i, p) \in Y_m(t_i), i = 0, \ldots, h \), where the \( Y_m(t_i) \)'s are connected sets of \( \mathbb{R}^m \). The SM-PE problem is formulated as finding the set of parameter vectors \( P \subseteq \mathbb{R}^p \) such that the arising trajectories hit all the output data sets, i.e.,

\[ p^* \in P \iff y(t_i, p^*) \in Y_m(t_i), \quad \forall i = 0, \ldots, h. \]

\( P \) is called the feasible parameter set (FPS). SM-PE problems are generally solved with a branch and bound algorithm that enumerates candidate solutions thanks to a rooted tree and assumes the full parameter space as the root set. At every node, the set of trajectories arising from the parameter set considered is checked for consistency against the measurements and labelled feasible, unfeasible or undetermined. Unfeasible sets are rejected while undetermined sets are split and checked in turn until the diameter of the candidate solution set is smaller than or equal to a given threshold \( \varepsilon \) provided by the user. Here \( \varepsilon \) is the precision threshold (or the precision for short) of the SM-PE algorithm. The SIVIA (set inversion via interval analysis) algorithm (Jaulin and Walter, 1993) can be cited to exemplify the above principles (branch and bound (bisection) and interval analysis). The number of bisections to be performed is generally prohibitive. Hence, recent algorithms take advantage of constraint propagation techniques to reduce the width of the boxes to be checked. In this context, the model is interpreted as the set of constraints of a constraint satisfaction problem (CSP). For solving such a CSP, various types of the so-called contractors can be used (Chabert and Jaulin, 2009).

It should be noted that such algorithms are anytime by nature, i.e., they provide a guaranteed solution independently of the stopping time, which redeems in some way their exponential complexity. The returned solution is an overestimation of the FPS given by the convex union of the candidates that have been labelled feasible and undetermined. Interestingly, the convex union may consist of one set or more (cf. the work of Jaulin et al. (2001) for several variants). In the following, we refer to the SM-PE algorithm as to a generic SM-PE algorithm based on these principles.

When considering an SM-PE problem, one would like to know beforehand whether or not \( P \) is reduced to one single connected set. In much the same may as for classical parameter estimation, this property indicates whether the problem is mathematically well-posed.

**Definition 7.** An SM-PE problem is said to be sound if \( P \subseteq U_P \) is reduced to one single connected set. In this case, \( P \) is also said to be sound.

Given an SM-PE algorithm with precision threshold \( \varepsilon \), we denote by \( P_\varepsilon \) the solution set. Then it is important to know the properties of \( P_\varepsilon \) in relation to \( P \).

**Definition 8.** Assume that \( P \) is equal to the union of \( \kappa \geq 1 \) mutually disjoint connected sets. Then the solution set \( P_\varepsilon \) is said to be \( \varepsilon \)-consistent if \( P_\varepsilon \) is equal to the union of \( \kappa_\varepsilon \) mutually disjoint connected sets and \( \kappa_\varepsilon = \kappa \).

Overdetermination and algorithm precision result in \( P_\varepsilon \) overestimating \( P \), which may imply \( \kappa_\varepsilon < \kappa \). In this case, at least one of the sets composing \( P_\varepsilon \) includes several sets composing \( P \). \( \varepsilon \)-consistency is analyzed in Section 6.

### 5. Soundness

#### 5.1. Conditions for soundness.

**Proposition 5.** Consider the system \( \Gamma \) and assume that the set \( P \subseteq U_P \) is the FPS of an SM-PE problem for \( \Gamma \). Then \( P \) is sound if and only if \( P \) is globally SM-identifiable.

**Proof.** By definition, if \( P \) is a globally SM-identifiable set, those of \( \Gamma \) arising from \( P \) are different from the trajectories arising from the complement \( P^c = U_\Gamma \setminus P \). In addition, \( P \) is connected, hence \( P \) is sound. Conversely, if \( P \) is sound, by definition it is globally SM-identifiable. 

In addition to being SM-identifiable, assume that \( P \) is \( \mu \)-SM-identifiable for \( \Gamma \). In this case, it is interesting to note that \( P \) preserves soundness when the bounded error corrupting the output data is getting smaller and smaller. In this case, \( P \) is said to be \( \mu \)-sound. This is stated by the following result.

**Proposition 6.** Given the output data sets \( Y_m(t_i), i = 1, \ldots, h \), assume that \( P \) is sound. Then, if \( P \) is \( \mu \)-SM-identifiable, for \( \Gamma \), the FPS of the same problem with contracted output data sets \( \mu_i(Y_m(t_i)), i = 1, \ldots, h \), where the \( \mu_i \)'s are contractions, is also sound.

**Proof.** This proof uses Proposition 5. The result simply comes from the fact that if \( P \) is \( \mu \)-SM-identifiable for \( \Gamma \), then \( P \) is obviously SM-identifiable, and for all \( P \subseteq P \), \( P \) is also SM-identifiable. 

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C. Jauberthie et al.

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5.2. Example. Consider the model
\[
\begin{align*}
\dot{x}_1 &= (p_1 + 2(1 - p_2) \cos(p_1))x_1^2 + (1 - p_2)x_2, \\
\dot{x}_2 &= \sin(p_1)x_1, \\
y &= x_1,
\end{align*}
\]
where \((p_1, p_2) \in [-1, 4] \times [0.1/10] = \mathcal{U}_P.\)

By setting \(c_1 = \sin(p_1),\) with the elimination order \(\{c_1, p_2\} < \{y\} < \{x_1, x_2\},\) the Rosenfeld–Groebner algorithm gives the following differential polynomial:
\[
R(y, u) = \dot{y} - 2(p_1 + 2(1 - p_2) \cos(p_1))yy - (1 - p_2) \sin(p_1)y.
\]
In that case, the functional determinant is reduced to \(\Delta R(y) = \det(\dot{y}, y) = -y^2 \dot{y}\) and is not identically equal to 0.

In order to consider the initial condition, the function \(\phi: (p_1, p_2) \to ((p_1 + 2(1 - p_2) \cos(p_1)), (1 - p_2) \sin(p_1))\) has to be studied. By using the algorithm proposed in Ravanbod et al. (2014), Fig. (right) is obtained. \(\mathcal{U}_P = [-1, 4] \times [0.1/10]\) has been partitioned in two domains: a domain on which the function \(\phi\) is partially injective and hence corresponding to \(\mu\)-SM-identifiable subsets in grey color in the figure and two subsets in white color, each of them producing the same image. If a parameter estimation problem is formulated such that the FPS \(\mathcal{P}\) is in \(\mathcal{U}_P,\) we can now decide whether or not \(\mathcal{P}\) is sound. Indeed, if the inverse image of the trajectories hitting the output data sets entirely lies in a \(\mu\)-SM-identifiable subset, then \(\mathcal{P}\) is sound. On the contrary, \(\mathcal{P}\) is unsound.

Fig. 1. Set of diameter \(\varepsilon\) interpolated between \(\mathcal{P}_1, \mathcal{P}_2\) and their undistinguishability neighborhoods (left), and the partition of the parameter domain: the grey color subsets are \(\mu\)-SM-identifiable (right).

6. \(\varepsilon\)-consistency

6.1. Conditions for \(\varepsilon\)-consistency. As expressed in Definition 8, \(\varepsilon\)-consistency is a property of the solution set \(\mathcal{P}_\varepsilon\) returned by the SM-PE algorithm with a specified precision threshold \(\varepsilon.\) Among the problems that impact \(\varepsilon\)-consistency, two are analyzed in this paper:

- the SM-PE algorithm may not be able to separate the mutually disjoint connected sets forming \(\mathcal{P}\) by testing topologically relevant candidate solution sets;
- trajectories arising from solution parameters may not be distinguishable from trajectories arising from nonsolution parameters, given the precision of the sensors.

**Proposition 7.** If the FPS \(\mathcal{P}\) is sound, then the solution set \(\mathcal{P}_\varepsilon\) is \(\varepsilon\)-consistent for any \(\varepsilon.\)

*Proof.* If \(\mathcal{P}\) is sound, it is reduced to one single connected set. Then, from the principle of branch and bound algorithms, the solution set \(\mathcal{P}_\varepsilon\) is also reduced to one single connected set although it may be an overestimation of \(\mathcal{P}.\)

Let us now assume that \(\mathcal{P}\) is unsound and consists of \(\kappa\) mutually disjoint connected sets, say \(\mathcal{P}_i, i = 1, \ldots, \kappa.\) The fact that the SM-PE algorithm is able to separate the \(\mathcal{P}_i\)'s is a topological problem involving the distance between the \(\mathcal{P}_i\)'s and the diameter of the smallest candidate solution sets considered by the branch and bound SM-PE algorithm.

**Proposition 8.** If \(\mathcal{P}\) is unsound and consists of \(\kappa\) mutually disjoint connected sets \(\mathcal{P}_i, i = 1, \ldots, \kappa,\) then a necessary condition for the solution set \(\mathcal{P}_\varepsilon\) returned by the SM-PE algorithm with precision threshold \(\varepsilon\) to be \(\varepsilon\)-consistent is that \(d(\mathcal{P}_i, \mathcal{P}_j) > \varepsilon, \forall i, j = 1, \ldots, \kappa, i \neq j.\)

*Proof.* With no loss of generality, consider two mutually disjoint connected sets \(\mathcal{P}_1, \mathcal{P}_2.\) The successive partitions of the parameter space arising from the branch and bound procedure provide candidate solution sets whose diameter is greater than or equal to \(\varepsilon.\) \(\mathcal{P}_1\) and \(\mathcal{P}_2\) are separable if any candidate solution set of diameter \(\varepsilon\) can be interposed anywhere between the two sets, in particular just where the two sets are closest. Only in such a case, i.e., if \(d(\mathcal{P}_1, \mathcal{P}_2) > \varepsilon,\) can the interposed candidate solution be labelled unfeasible, hence rejected by the algorithm, and \(\mathcal{P}_\varepsilon\) composed of two mutually disjoint sets.

*Notice that IAVIA gives no information about these two domains: they are labelled undetermined.*
analytic on $M$. $K$ Lipschitz constants are respectively denoted $(iii)$ if the initial conditions depend on $h,x$. Assume that the assumptions $(i)-(iii)$ are satisfied. Proposition 9. If the initial conditions do not depend on $p$, then it is sufficient to set $K_{x_0,p} = 0$ in the previous proof.

From this result, one can determine the minimal distance between two parameter vectors $p$ and $\bar{p}$ for which the trajectories $y(t,p)$ and $y(t,\bar{p})$ are ensured to be distant by $\lambda$. Applying the Lipschitz result of $y(t,p)$ with respect to $p$, the following corollary is deduced.
Corollary 3. Let us consider two trajectories $y(\cdot, p)$ and $y(\cdot, \hat{p})$ arising from $\Gamma$. Then, under conditions (i)–(iii), if
\[ \exists t \in [t_0, T], \|y(t, p) - y(t, \hat{p})\|_\infty > \lambda \]
then
\[ \|p - \hat{p}\| > \frac{\lambda}{K_{y,p}}. \]

Proof. Since $y$ is Lipschitz continuous according to the parameter vector $p$, we get
\[ \lambda < \|y(t, p) - y(t, \hat{p})\|_\infty < K_{y,p}\|p - \hat{p}\|. \]
which implies
\[ \|p - \hat{p}\| > \frac{\lambda}{K_{y,p}}. \]

This result means that the $P_i$’s forming $\mathcal{P}$ are surrounded by a neighborhood that may generate trajectories that are not distinguishable from those arising from their inside parameters.

Putting together the results of Proposition 8 and Corollary 3, we obtain the following condition for $\varepsilon$-consistency.

Proposition 10. Consider the system $\Gamma$ along with the assumptions (i)–(iii) and assume that $P$ is unsound and consists of $\kappa$ mutually disjoint connected sets $\mathcal{P}_i, i = 1, \ldots, \kappa$. If the solution set $\mathcal{P}_e$ returned by the SM-PE algorithm with precision threshold $\varepsilon$ is $\varepsilon$-consistent, then
\[ d(\mathcal{P}_i, \mathcal{P}_j) > \varepsilon + \frac{2\lambda}{K_{y,p}}, \quad \forall i, j = 1, \ldots, \kappa, \]
where $\lambda$ is the precision of the sensors.

Proof. With no loss of generality, assume that $\mathcal{P}$ consists of two mutually disjoint connected sets $\mathcal{P}_1$ and $\mathcal{P}_2$. Denote by $\mathcal{N}_{\mathcal{P}_1}$ and $\mathcal{N}_{\mathcal{P}_2}$ the undistinguishability neighborhoods of $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. The sets to be separated by the SM-PE algorithm are hence $\mathcal{P}_1 \cup \mathcal{N}_{\mathcal{P}_1}$ and $\mathcal{P}_2 \cup \mathcal{N}_{\mathcal{P}_2}$. Then Proposition 8 applied to these sets implies $d(\mathcal{P}_1 \cup \mathcal{N}_{\mathcal{P}_1}, \mathcal{P}_2 \cup \mathcal{N}_{\mathcal{P}_2}) > \varepsilon$. The characterization of $\mathcal{N}_{\mathcal{P}_1}$ and $\mathcal{N}_{\mathcal{P}_2}$ provided by Corollary 3 then implies $d(\mathcal{P}_1, \mathcal{P}_2) > \varepsilon + 2\lambda/K_{y,p}$ as illustrated in Fig. 1 (left).

Remark 2. The converse is also true if the inclusion function $P \rightarrow \mathcal{Y}(P)$ used by the SM-PE algorithm to predict the set of trajectories arising from a given candidate parameter set $P$ is such that $|\mathcal{Y}(P)| = \mathcal{Y}(P)$, which is rarely the case.

6.2. Example. Consider the following example defined on $[0, T]$:
\[
\begin{align*}
\dot{x}_1(t, p) &= \cos(p)x_2(t, p), \quad x_1(0) = (\frac{1}{\sqrt{2}} + 1)/2, \\
\dot{x}_2(t, p) &= -x_1(t, p)\cos(p) + (1 - x_1(t, p)^2)2x_2(t, p), \quad x_2(0) = 0, \\
y(t, p) &= x_1(t, p).
\end{align*}
\]
(15)
The functions $f$ and $h$ are defined by
\[ f(x, p) = \cos(p)x_2, \quad h(x, p) = x_1 \]
and $h(x, p) = x_1$ where $x = (x_1, x_2)$ and $T$ denotes the transpose of the considered vector. The solution $(x_1(t), x_2(t))^T$ remains in the ring $R$ defined by the two circles centered at $(0, 0)$ with radii $1/\sqrt{2}$ and 1. Indeed, we have
\[
\frac{d}{dt}(\frac{x_1^2 + x_2^2}{2}) = x_1\frac{dx_1}{dt} + x_2\frac{dx_2}{dt} = (1 - x_1^2 - 2x_2^2)x_2.
\]
Since $1 - x_1^2 - 2x_2^2$ is positive for $x_1^2 + x_2^2 < 1/2$ and negative for $x_1^2 + x_2^2 > 1$, $x_1^2 + x_2^2$ increases when $x_1^2 + x_2^2 > 1/2$ and decreases when $x_1^2 + x_2^2 > 1$. One can conclude that, according to the initial condition, the solution remains in the ring $R$.

The following step consists in finding the Lipschitz constants. Let us consider $z = (z_1, z_2) \in R$. Clearly, $K_{h,p} = 0$, $K_{h,x} = 1$ and $K_{f,p} = 1$ since $\|x\| = \|(x_1, x_2)^T\| < 1$. For $K_{f,x}$, by reordering the terms and adding $x_1^2z_2 - x_1^2z_2$ in line 3, we get
\[
\begin{align*}
\|f(x, p) - f(z, p)\|_1 &\leq |\cos(p)|(|x_2 - z_2| + |x_1 - z_1|) \\
&+ |x_2 - z_2| + (|x_1 - 2x_2^2|)x_2 - (z_1^2 - 2z_2^2)z_2 | \\
&\leq (|x_1 - z_1| + 2|x_2 - z_2|) \\
&+ |x_1^2z_2 - x_1^2z_2 + x_1^2z_2 \\
&- z_1^2z_2 - 2(z_2^2 - z_2^2) \\
&\leq (|x_1 - z_1| + 2|x_2 - z_2|) + x_1^2|x_2 - z_2| \\
&+ |z_2||x_1^2 - z_1^2| + 2|x_2^3 - z_2^3|.
\end{align*}
\]
(16)
Since $|x_1| < 1$, $|z_1| < 1$, we deduce, on the one hand, that $|x_1^2 - z_1^2| \leq |x_1 - z_1|(|x_1 + |z_1|) \leq 2|x_1 - z_1|$, and on the other that
\[
|z_2^3 - z_2^3| = |x_2 - z_2||z_2^2 + 2x_2z_2 + x_2^2| \\
\leq 4|x_2 - z_2|.
\]
Hence
\[
\begin{align*}
\|f(x, p) - f(z, p)\|_1 &\leq (|x_1 - z_1| + 2|x_2 - z_2|) \\
&+ |x_2 - z_2| + 2|x_1 - z_1| \\
&+ 8|x_2 - z_2| \leq 11\|x - z\|_1.
\end{align*}
\]
(17)
Using the equivalence between norms 1 and maximum norm, we get
\[ \|f(x, p) - f(z, p)\| \leq 22\|x - z\|. \]

Hence \( K_{f,p} = 22 \) and from Proposition \[9\] the Lipschitz constant \( K_{y,p} \) is equal to \( T e^{22T} \). Taking \( [0, T] = [0, 1] \), we get sensor precision \( \lambda = 0.01 \), and SM-PE algorithm precision threshold \( \varepsilon = 0.001 \). By Proposition \[10\] the solution set is \( \varepsilon \)-consistent, which implies that
\[ d(P_i, P_j) > \varepsilon + \frac{2\lambda}{K_{y,p}} \]
\[ \simeq 0.001 + 5.579 \cdot 10^{-12} \]
\[ \simeq 0.001, \quad \forall i, j = 1, \ldots, \kappa. \]

In this example, the SM-PE algorithm precision is dominant over the sensor precision with respect to \( \varepsilon \)-consistency.

7. Discussion and conclusions

This paper casts identifiability in an SM framework and relates the properties introduced by Jauberthie et al. (2011; 2013), namely, SM-\( \mu \)-SM-\( \varepsilon \)-SM-identifiability, to the properties of SM-PE problems. Soundness and \( \varepsilon \)-consistency are proposed to characterize an SM-PE problem. Soundness is a theoretical property that assesses that the SM-PE problem is well-posed. Note that \( \varepsilon \)-consistency guarantees that the structure of the FPS is well reflected in the solution returned by the SM-PE algorithm.

SM-\( \mu \)-SM-\( \varepsilon \)-SM-identifiability is compared with related properties existing in the literature, in particular partial injectivity. The differential algebra based method proposed to check these properties leads to checking partial-injectivity and a newly introduced property named partial-R-injectivity. The algorithm proposed for this (Ravanbod et al., 2014) remains of exponential complexity like many interval-based algorithms, but it is still useful for medium-size problems.

Accordingly, \( \varepsilon \)-consistency is a complex property for which only necessary conditions are provided. It is impacted by several features of the SM-PE problem, including sensor precision and the overestimation involved in the computation of the image of a parameter set. Evaluating this overestimation and how it impacts \( \varepsilon \)-consistency remains an open problem.

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References


Set-membership identifiability of nonlinear models and related parameter estimation properties


Carine Jauberthie has been an associate professor at the University Paul Sabatier of Toulouse (France) since 2005. She is a researcher at CNRS Laboratoire d’Analyse et d’Architecture des Systèmes (LAAS) in the Diagnosis and Supervisory Control (DISCO) Research Team. She obtained a Ph.D. degree in 2002 in applied mathematics, specializing in system control, from the University of Technology of Compiègne, France, at the ONERA Center of Lille in collaboration with the Laboratory of Applied Mathematics. Her research interests concern fault detection and diagnosis as well as the analysis of related properties, including identifiability and diagnosability of nonlinear dynamical systems with mixed uncertainties.

Louise Travé-Massuyés holds the position of the research director at Laboratoire d’Analyse et d’Architecture des Systèmes, Centre National de la Recherche Scientifique (LAAS-CNRS), Toulouse, France, and was the head of the Diagnosis and Supervisory Control (DISCO) team from 1994 to 2015. She graduated in control engineering from Institut National des Sciences Appliqués (INSA), Toulouse, France, in 1982 and received the Ph.D. degree from INSA in 1984. Her research interests are in dynamic systems supervision, with special focus on qualitative and model-based methods, as well as data mining. She has been particularly active in bridging the AI and control engineering diagnosis fields as the leader of the BRIDGE Task Group of the MONET European Network of Excellence. She is the coordinator of the Maintenance & Diagnosis Strategic Field within the Aerospace Valley World Competitiveness Cluster, and serves as the contact evaluator for the French Research Funding Agency. She serves on editorial boards of artificial intelligence journals. She is also a member of the IFAC Safe-process Technical Committee.

Nathalie Verdière has been an associate professor at the University of Le Havre (France) since 2006. She is a researcher at the Laboratory of Applied Mathematics of Le Havre in the Dynamical Systems Research Team. She obtained a Ph.D. degree in 2005 in applied mathematics (specialization in control systems) from the University of Technology of Compiègne, France. From 2005 to 2006, she was a temporary associate professor at the University of Technology of Compiègne. Her research interests concern bifurcations, identifiability analysis, parameter estimation, fault detection and diagnosis for models described by ordinary or partial differential equations.

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