POSITIVITY AND LINEARIZATION OF A CLASS OF NONLINEAR CONTINUOUS–TIME SYSTEMS BY STATE FEEDBACKS

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The positivity and linearization of a class of nonlinear continuous-time system by nonlinear state feedbacks are addressed. Necessary and sufficient conditions for the positivity of the class of nonlinear systems are established. A method for linearization of nonlinear systems by nonlinear state feedbacks is presented. It is shown that by a suitable choice of the state feedback it is possible to obtain an asymptotically stable and controllable linear system, and if the closed-loop system is positive then it is unstable.

Keywords: positive, nonlinear, system, linearization, state feedback.

1. Introduction

In positive systems inputs, state variables and outputs take only nonnegative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc. Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced.


In this paper the positivity and linearization of a class of nonlinear continuous-time systems by nonlinear state feedbacks will be addressed. The paper is organized as follows. In Section 2, necessary and sufficient conditions for the positivity of a class of nonlinear systems are established. Linearization of the nonlinear system by a nonlinear state feedback is addressed in Section 3. An example illustrating the discussion is given in Section 4. Concluding remarks are presented in Section 5.

The following notation will be used: \( \mathbb{R} \), the set of real numbers; \( \mathbb{R}^{n \times m} \), the set of \( n \times m \) real matrices and \( \mathbb{R}^{n} = \mathbb{R}^{n \times 1} \); \( \mathbb{R}_{+}^{n \times m} \), the set of \( n \times m \) matrices with nonnegative entries and \( \mathbb{R}_{+}^{n} = \mathbb{R}_{+}^{n \times 1} \); \( M_n \), the set of \( n \times n \) Metzler matrices (with nonnegative off-diagonal entries); \( I_n \), the \( n \times n \) identity matrix.

2. Positivity of nonlinear systems

Consider the nonlinear system

\[ \dot{x} = Ax + f(x) + Bu, \]  

(1)
where

\[
x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix},
\]

\[
f(x) = \begin{bmatrix} f_1(x_1) \\ f_2(x_1, x_2) \\ \vdots \\ f_n(x_1, \ldots, x_n) \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},
\]

and every \(u)\) is a Meltzer matrix,

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\[\begin{aligned}
x &= x(t) \in \mathbb{R}^n, \quad u = u(t) \in \mathbb{R} & \quad \text{are the state vector and the input vector, respectively.}
\end{aligned}\]

It is assumed that the functions \(f_k(x_1, \ldots, x_k), k = 1, 2, \ldots, n,\) are continuously differentiable for all their arguments.

**Definition 1.** The nonlinear system (1) is called (internally) positive if \(x(t) \in \mathbb{R}_+^n\) for all \(x(0) \in \mathbb{R}^n, t \geq 0\) and every \(u(t) \in \mathbb{R}_+, t \geq 0\).

**Theorem 1.** The nonlinear system (1) is positive if and only if

\[
f_k(\bar{x}) \in \mathbb{R}_+ \text{ for } \bar{x} = [x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_k(t)]^T \in \mathbb{R}_+, \quad j = 1, 2, \ldots, k \quad \text{and } u(t) \in \mathbb{R}_+, t \geq 0.
\]

**Proof.** For given \(f(x)\), the solution of (1) has the form

\[
x(t) = e^{Ax(0)} + \int_0^t e^{A(t-\tau)}[f(x(\tau)) + Bu(\tau)] \, d\tau. \tag{4}
\]

The linear system obtained from (1) for \(f_k(x_1, \ldots, x_k) = 0, k = 1, 2, \ldots, n,\) is positive since the matrix \(A\) is a Meltzer matrix, \(B \in \mathbb{R}^n_+\).

Using the well-known Picard method, the \(k\)-approximation of the solution of (1) can be found from the formula

\[
x_{k+1}(t) = e^{Ax(0)} + \int_0^t e^{A(t-\tau)}[f(x_k(\tau)) + Bu(\tau)] \, d\tau \tag{5}
\]

for \(k = 1, 2, \ldots, n - 1\).

The Lipschitz conditions for (1) are satisfied since, by assumption, the functions \(f_k(x_1, \ldots, x_k), k = 1, 2, \ldots, n,\) are continuously differentiable. Using the Picard method, it is easy to show that Eqn. (1) has nonnegative solution \(x(t) \in \mathbb{R}_+^n, t \geq 0\) if and only if the conditions (3) are satisfied.

The proof can be also accomplished using the method presented by Malesza and Respondek (2007).

### 3. Linearization by state feedbacks

For the nonlinear system (1), we introduce the following new state variables (the components of the new state vector \(z = [z_1 \ldots z_n]^T\)):

\[
z_1 = x_1,
\]

\[
z_2 = x_2 + f_1(x_1),
\]

\[
z_3 = x_3 + f_2(x_1, x_2) + \frac{\partial f_1}{\partial x_1} [x_2 + f_1(x_1)]
\]

\[
= x_3 + f_2(x_1, x_2),
\]

\[
z_4 = x_4 + f_3(x_1, x_2, x_3) + \frac{\partial f_2}{\partial x_2} [x_3 + f_2(x_1, x_2)]
\]

\[
+ \frac{\partial f_2}{\partial x_2} [x_2 + f_1(x_1)]
\]

\[
= x_4 + f_3(x_1, x_2, x_3),
\]

\[
\vdots
\]

\[
z_n = x_n + f_{n-1}(x_1, \ldots, x_{n-1}).
\]

The relations (6) can be written shortly as \(z = \phi(x)\).

From (6), we have

\[
x_1 = z_1,
\]

\[
x_2 = z_2 - f_1(z_1),
\]

\[
x_3 = z_3 - f_2(z_1, z_2),
\]

\[
\vdots
\]

\[
x_n = z_n - f_{n-1}(z_1, \ldots, z_{n-1}).
\]

This can be briefly expressed as \(x = \phi^{-1}(z)\).

The nonlinear system (1) in the new state variables (6) has the form

\[
f(z) = e^{A_0} x(z_0) + \int_0^t e^{A_0(t-\tau)}[f(z(\tau)) + B u(\tau)] \, d\tau.
\]

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\[
x_2 = z_2 - f_1(z_1),
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x_3 = z_3 - f_2(z_1, z_2),
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x_n = z_n - f_{n-1}(z_1, \ldots, z_{n-1}).
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The nonlinear system (1) in the new state variables (6) has the form

\[
f(z) = e^{A_0} x(z_0) + \int_0^t e^{A_0(t-\tau)}[f(z(\tau)) + B u(\tau)] \, d\tau.
\]
where
\[ v = u + g(x), \]
\[ g(x) = \sum_{i=0}^{n} a_i z_{i+1} \big|_{z=\phi(x)} + f_n(x_1, \ldots, x_n) \]
\[ + \frac{\partial f_{n-1}}{\partial x_1}[f_2 + f_1(x_1)] \]
\[ + \cdots + \frac{\partial f_{2}}{\partial x_{n-1}}[x_n + f_{n-1}(x_1, \ldots, x_{n-1})]. \]
(Eqn. (10))

Theorem 3. The nonlinear system (1) can be linearized by the nonlinear state feedback (12), so that the closed-loop system (10) for \( a_k = 0, k = 0, 1, \ldots, n-1 \), is positive but unstable.

4. Example
Consider the nonlinear system described by the equations
\[ \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 + x_1^2 \\ x_3 + x_1 x_2 \\ x_2 x_3 + u \end{bmatrix}, \quad x(0) \in \mathbb{R}^3. \] (14)

The system (14) is positive since Eqn. (14) satisfies the conditions (3) and \( u = u(t) \in \mathbb{R}_+, t \geq 0 \). In this case, the new state variables \( z_k, k = 1, 2, 3 \), are defined as follows:
\[ z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 + x_1^2 \\ x_3 + 3 x_1 x_2 + 2 x_1^3 \end{bmatrix} = \phi(x) \]
and
\[ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 - z_1^2 \\ z_3 - 3 z_1 z_2 + z_1^3 \end{bmatrix}. \] (16)

The nonlinear system (13) in the new state variables is described by Eqn. (17).

To linearize the nonlinear system (17), we apply the nonlinear state feedback (12) of the form
\[ u = v - g(z) = v - a_0 z_1 - a_1 z_2 - a_2 z_3 + z_1^5 + 2 z_1^4 - 4 z_1^3 z_2 
+ 3 z_1^2 z_2 + z_1^3 z_3 - 3 z_2^3 + 3 z_1 z_2^2 
- 3 z_1 z_3 - z_2 z_3, \]
and we obtain the linear system (10) with
\[ A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \] (19)

The linear system is controllable for all values of the coefficients \( a_k, k = 0, 1, 2 \), and it is asymptotically stable if and only if \( a_k > 0, k = 0, 1, 2, \) and \( a_1 a_2 > a_0 \).

The linear system (10) with (19) is positive if and only if \( a_k = 0, k = 0, 1 \) since in this case
\[ A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \in M_3. \] (20)

In this case the linear system is unstable.

5. Concluding remarks
The positivity and linearization of a class of nonlinear systems by nonlinear state feedbacks were addressed. Necessary and sufficient conditions for the positivity of the class of nonlinear systems (Theorem 1) were established. It was shown that the nonlinear systems can be linearized by nonlinear feedbacks, so that the
linear close-loop system is asymptotically stable and controllable (Theorem 2) and positive but unstable (Theorem 3). The discussion was illustrated by an example. An open problem is the extension of these deliberations to fractional nonlinear systems.

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References
the Polish Academy of Sciences in Rome. In 2004 he was elected an honorary member of the Hungarian Academy of Sciences. He was granted honorary doctorates by 13 universities. His research interests cover systems theory, especially singular multidimensional systems, positive multidimensional systems, singular positive 1D and 2D systems, as well as positive fractional 1D and 2D systems. He initiated research in the field of singular 2D, positive 2D and positive fractional linear systems. He published 28 books (8 in English) and over 1100 scientific papers. He also supervised 69 Ph.D. theses. He is the editor-in-chief of the Bulletin of the Polish Academy of Sciences: Technical Sciences and a member of editorial boards of ten international journals.

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